EE3054
Signals and Systems

Fourier Transform: Important Properties

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Some slides included are extracted from lecture presentations prepared by McClellan and Schafer

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LECTURE OBJECTIVES

- Basic properties of Fourier transforms
  - **Duality, Delay, Freq. Shifting, Scaling**
  - **Convolution** property
  - **Multiplication** property
  - **Differentiation** property
  - Freq. Response of Differential Equation System

Fourier Transform Defined

- For non-periodic signals

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis

Table of Fourier Transforms

$$x(t) = e^{-t}u(t) \quad \Leftrightarrow \quad X(j\omega) = \frac{1}{1 + j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \quad \Leftrightarrow \quad X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$

$$x(t) = \frac{\sin(\omega_b t)}{\pi t} \quad \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}$$

$$x(t) = \delta(t) \quad \Leftrightarrow \quad X(j\omega) = 1$$

$$x(t) = \delta(t - t_0) \quad \Leftrightarrow \quad X(j\omega) = e^{-j\omega t_0}$$

$$x(t) = e^{j\omega_c t} \quad \Leftrightarrow \quad X(j\omega) = 2\pi\delta(\omega - \omega_c)$$

$$x(t) = \cos(\omega_c t) \quad \Leftrightarrow \quad X(j\omega) = \pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c)$$

$$x(t) = \sin(\omega_c t) \quad \Leftrightarrow \quad X(j\omega) = -j\pi\delta(\omega - \omega_c) + j\pi\delta(\omega + \omega_c)$$

Duality of FT Pairs

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

If $\quad x(t) \Leftrightarrow g(\omega)$

Then $\quad g(t) \Leftrightarrow 2\pi x(-\omega)$

Fourier Transform of a General Periodic Signal

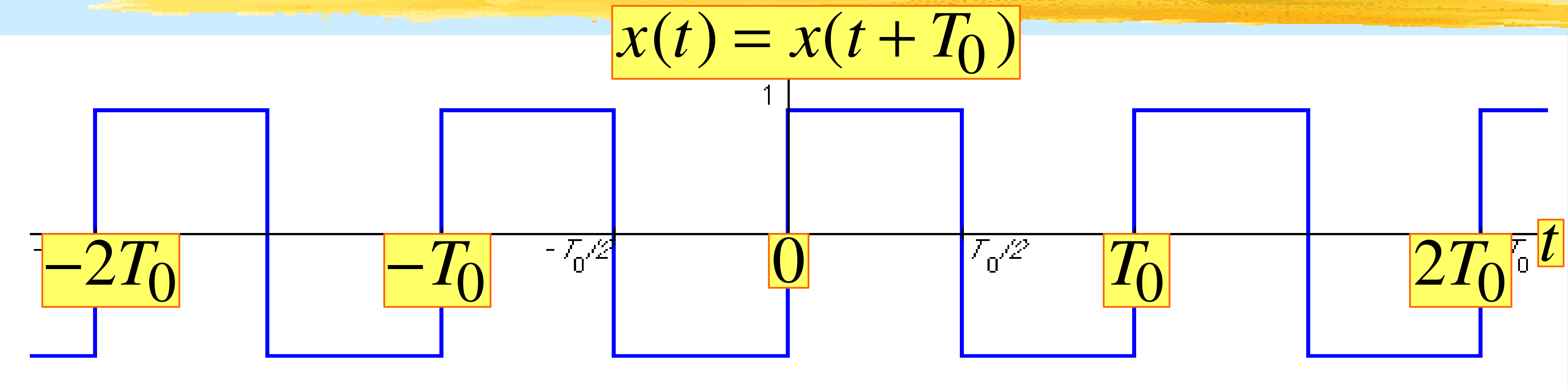
- If *x(t)* is periodic with period T_0 ,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-jk\omega_0 t} dt$$

$$\text{Therefore, since } \; e^{jk\omega_0 t} \Leftrightarrow 2\pi\delta(\omega - k\omega_0)$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

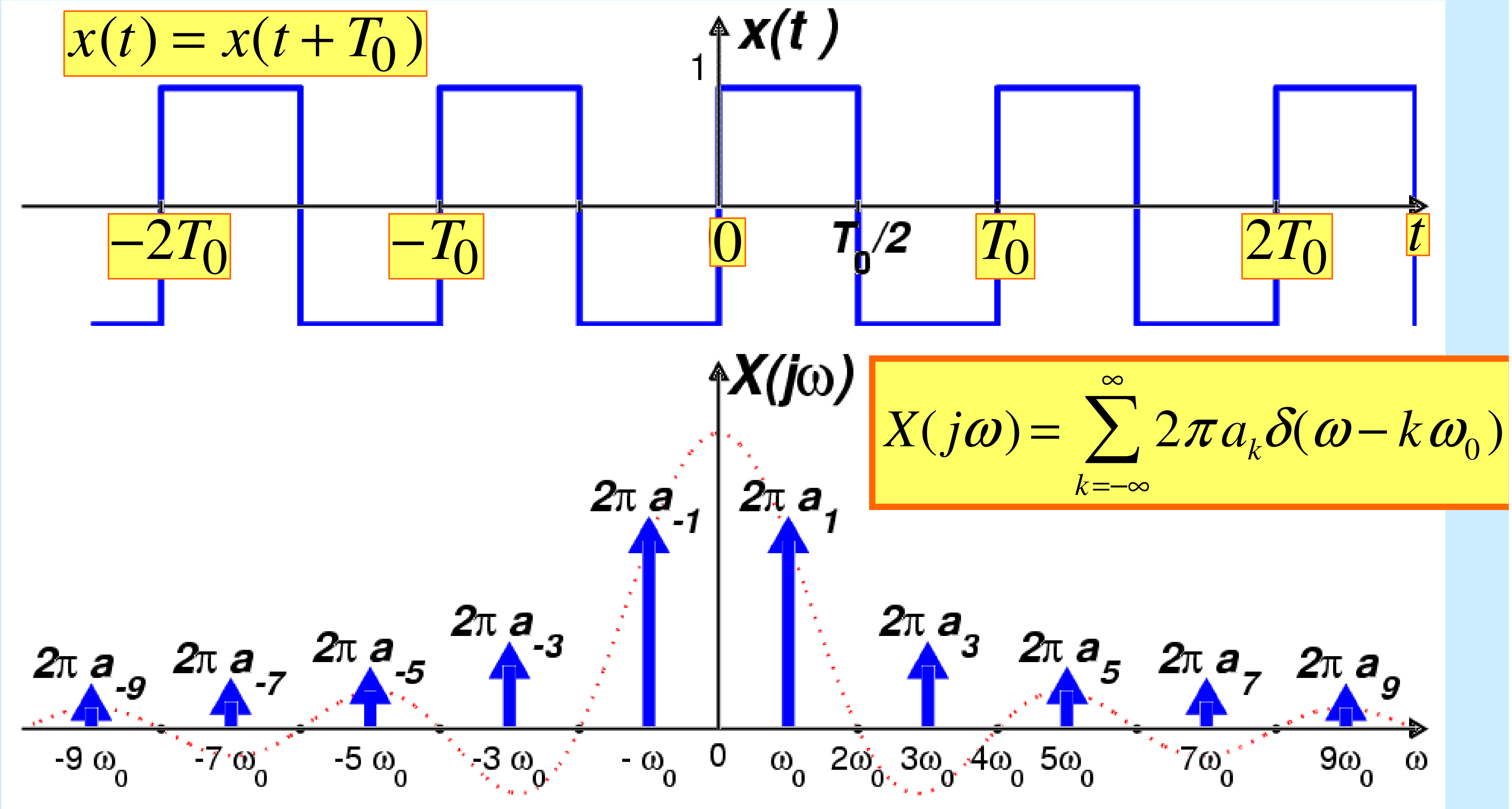
Square Wave Signal



$$a_k = \frac{1}{T_0}\int_0^{T_0/2}(1)e^{-j\omega_0 kt}dt + \frac{1}{T_0}\int_{T_0/2}^{T_0}(-1)e^{-j\omega_0 kt}dt$$

$$a_k = \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_0^{T_0/2} - \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_{T_0/2}^{T_0} = \frac{1-e^{-j\pi k}}{j\pi k}$$

Square Wave Fourier Transform



FT of Impulse Train

- The periodic impulse train is

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{n=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\omega_0 = 2\pi / T_0$$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) e^{-j\omega_0 t} dt = \frac{1}{T_0} \quad \text{for all } k$$

$$\therefore \quad P(j\omega) = \left(\frac{2\pi}{T_0}\right) \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$

Plot of impulse train in time and frequency

Table of Easy FT Properties

Linearity Property

$$a x_1(t) + b x_2(t) \Leftrightarrow a X_1(j\omega) + b X_2(j\omega)$$

Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

Duality

Frequency Shifting

$$x(t) e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Scaling

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j(\tfrac{\omega}{a}))$$

Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

$$\int_{-\infty}^{\infty} x(t - t_d) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega(\tau + t_d)} d\tau$$

$$= e^{-j\omega t_d} X(j\omega)$$

For example, $e^{-a(t-5)} u(t-5) \Leftrightarrow \dfrac{e^{-j\omega 5}}{a + j\omega}$

Multiply by e^jw0

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

$$\int_{-\infty}^{\infty} e^{j\omega_0 t} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-j(\omega - \omega_0)t} dt$$

$$= X(j(\omega - \omega_0))$$

Multiply by cos(w0)?

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

$$x(t)e^{-j\omega_0 t} \Leftrightarrow X(j(\omega + \omega_0))$$

$$x(t)\cos(\omega_0 t) = \frac{1}{2}\left(x(t)e^{j\omega_0 t} + x(t)e^{-j\omega_0 t}\right) \Leftrightarrow$$

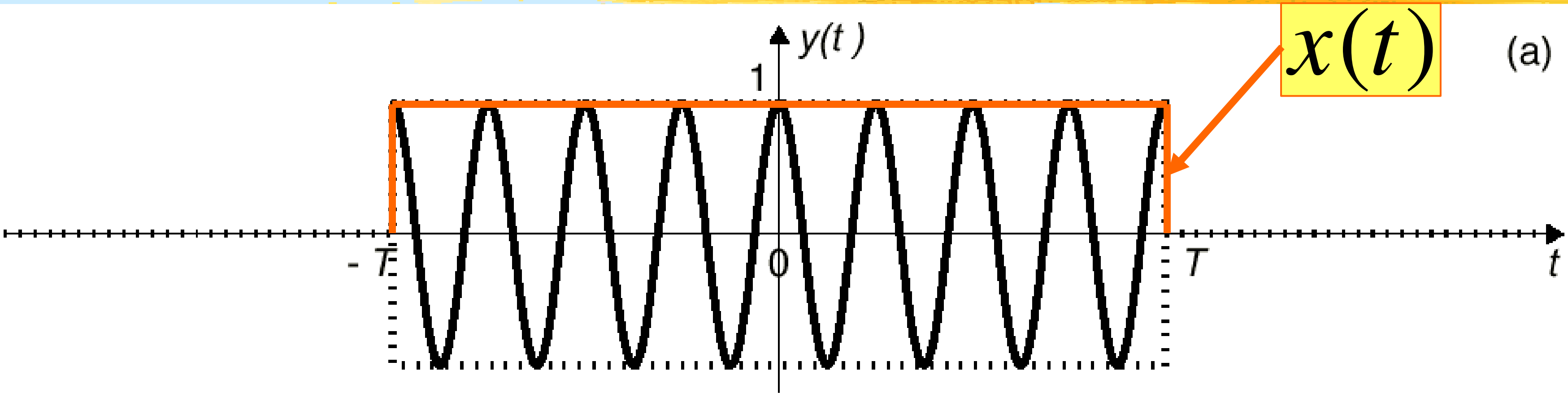
$$\frac{1}{2}\left(X(j(\omega - \omega_0)) + X(j(\omega + \omega_0))\right)$$

Shifting in frequency by multiply by cos() = (Amplitude Modulation)

- Illustrate the spectrum in class

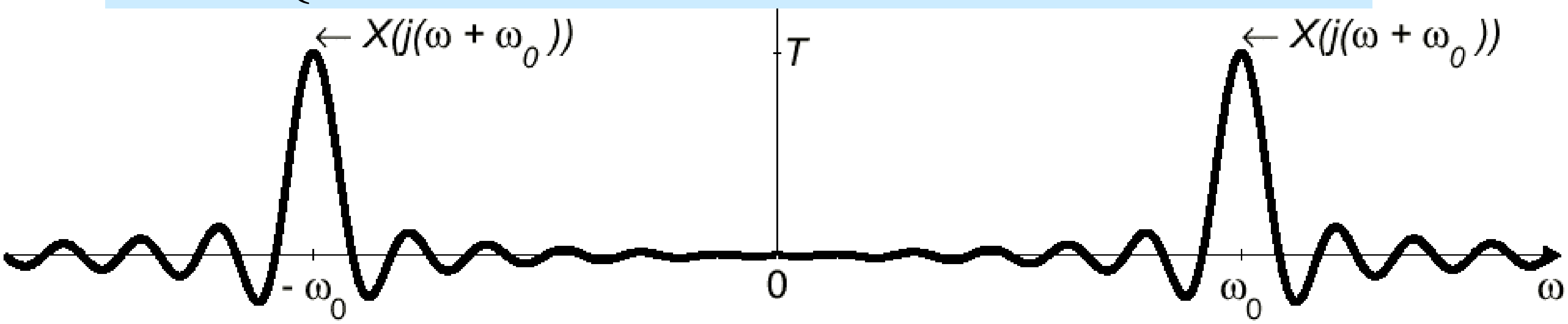
$$y(t) = x(t)\cos(\omega_0 t) \Leftrightarrow$$

$$Y(j\omega) = \tfrac{1}{2} X(j(\omega - \omega_0)) + \tfrac{1}{2} X(j(\omega + \omega_0))$$



$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

(b)



Another example

- x(t)=cos (w0 t)
- What is y(t)=x(t) * cos (w1 t)
  - Consider w1 >w0 and w1<w0
  - Verify by trigonometric identities

What about multiply by sin( )?

Scaling Property

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j\frac{\omega}{a})$$

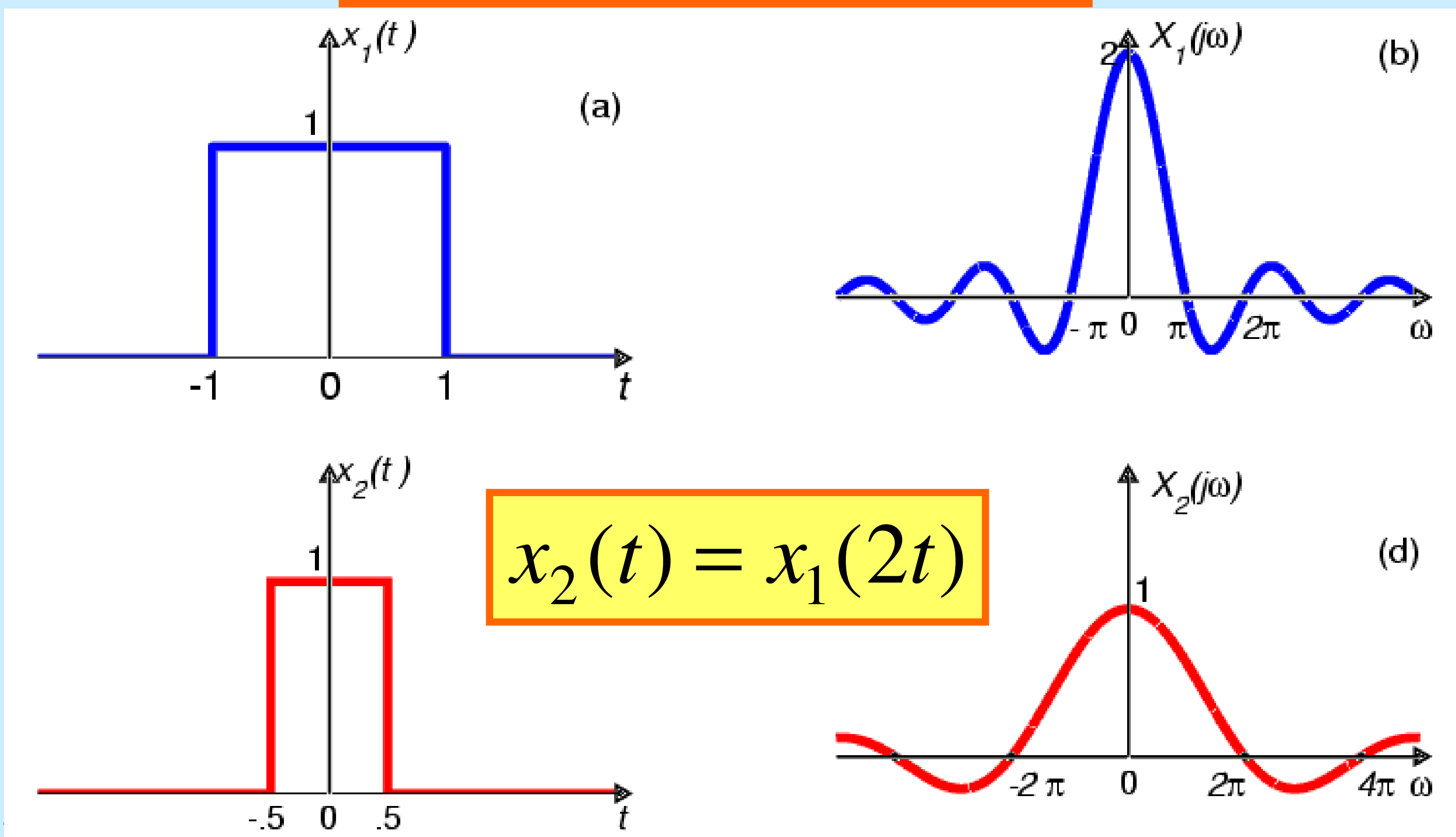
$$\int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega(\lambda/a)} \frac{d\lambda}{|a|}$$

$$= \frac{1}{|a|} X(j\frac{\omega}{a})$$

$x(2t)$ shrinks; $\frac{1}{2} X(j\frac{\omega}{2})$ expands

Scaling Property

$$x(at) \Leftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$



$$x_2(t) = x_1(2t)$$

Uncertainty Principle

- Try to make x(t) shorter
  - Then X(jω) will get wider
  - Narrow pulses have wide bandwidth
- Try to make X(jω) narrower
  - Then x(t) will have longer duration
- **Cannot simultaneously reduce time duration and bandwidth**

Table of Easy FT Properties

Linearity Property

$$ax_1(t) + bx_2(t) \Leftrightarrow aX_1(j\omega) + bX_2(j\omega)$$

Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

Frequency Shifting

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Scaling

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j(\frac{\omega}{a}))$$

Significant FT Properties

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

Duality

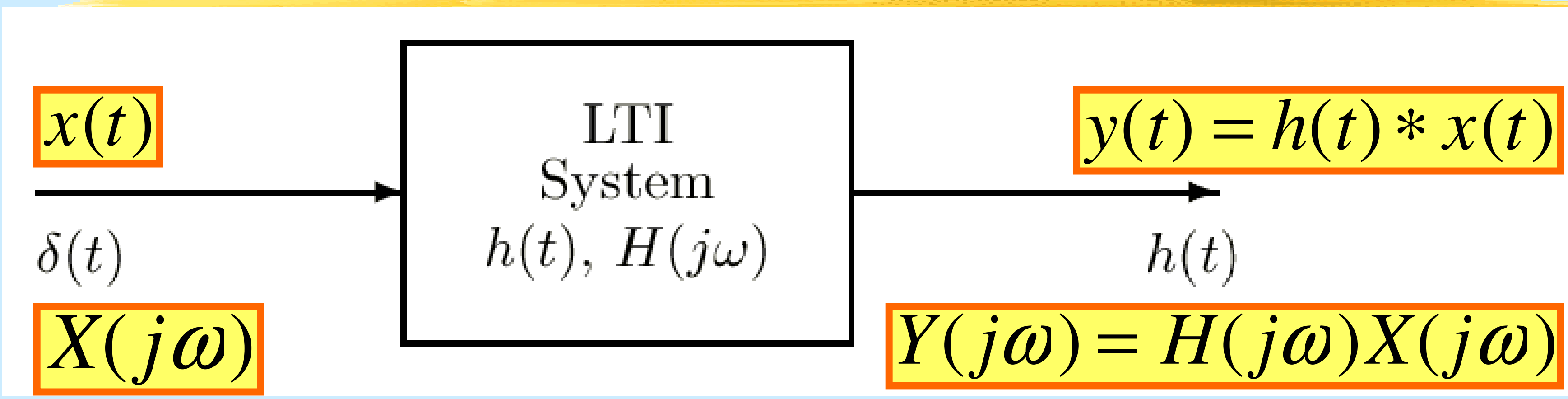
$$x(t)p(t) \Leftrightarrow \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Differentiation Property

$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega)X(j\omega)$$

Convolution Property



- Convolution in the time-domain

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

corresponds to **MULTIPLICATION** in the frequency-domain $Y(j\omega) = H(j\omega)X(j\omega)$

Proof (in class)

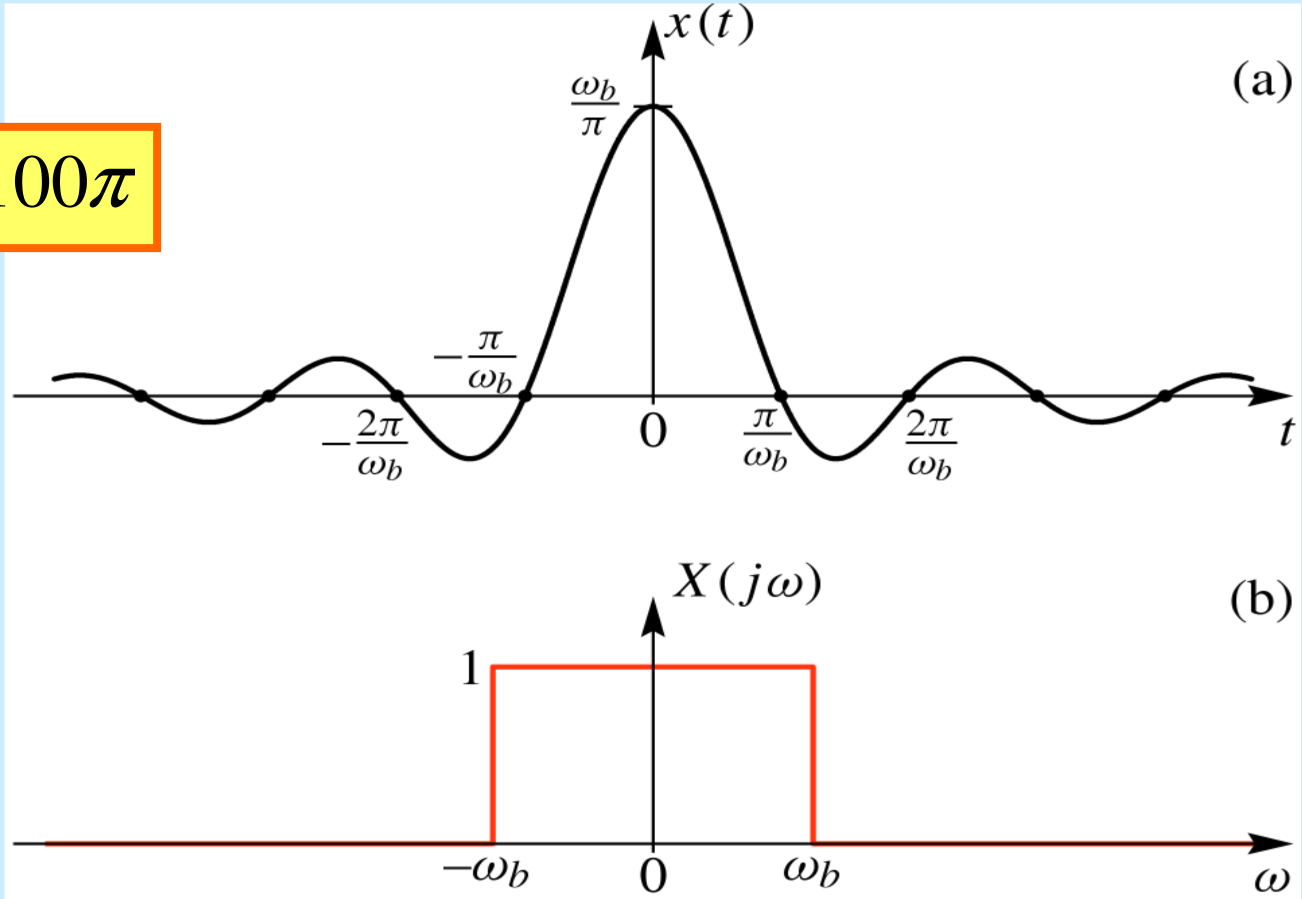
Convolution Example

- Bandlimited **Input** Signal
  - “sinc” function
- Ideal LPF (Lowpass Filter)
  - **h(t)** is a “sinc”
- **Output** is Bandlimited
  - Convolve “sincs”

Ideally Bandlimited Signal

$$x(t) = \frac{\sin(100\pi t)}{\pi t} \quad \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < 100\pi \\ 0 & |\omega| > 100\pi \end{cases}$$

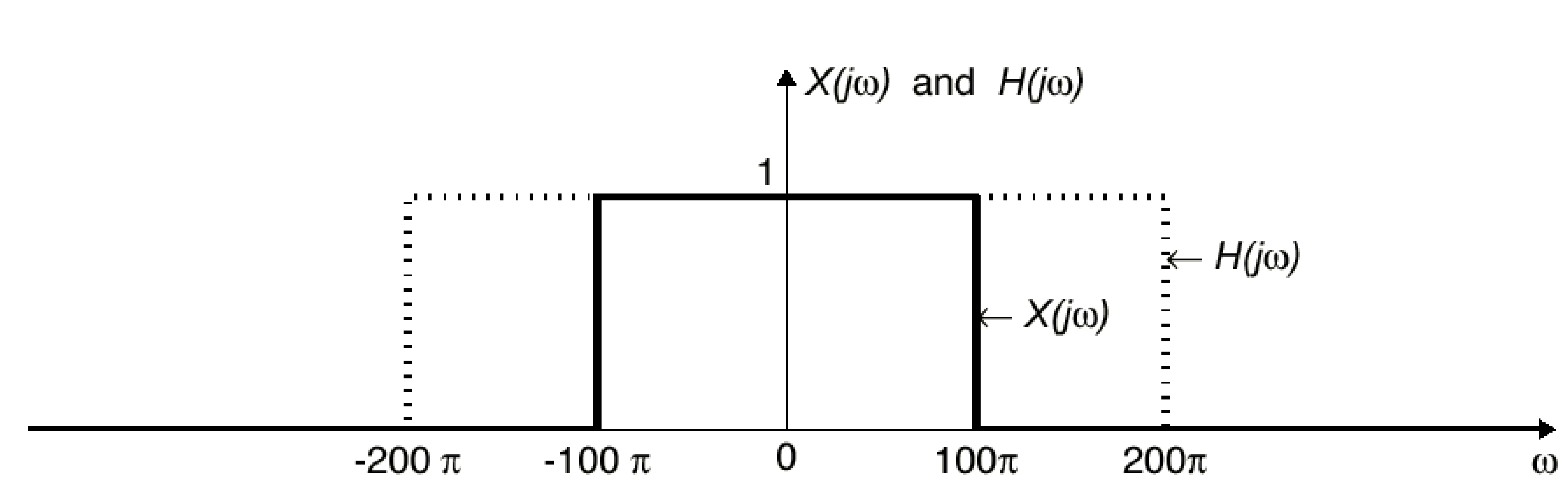
$\omega_b = 100\pi$



Ex: x(t) and y(t) are both sinc

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

$$\frac{\sin(100\pi t)}{\pi t} * \frac{\sin(200\pi t)}{\pi t} = \frac{\sin(100\pi t)}{\pi t}$$

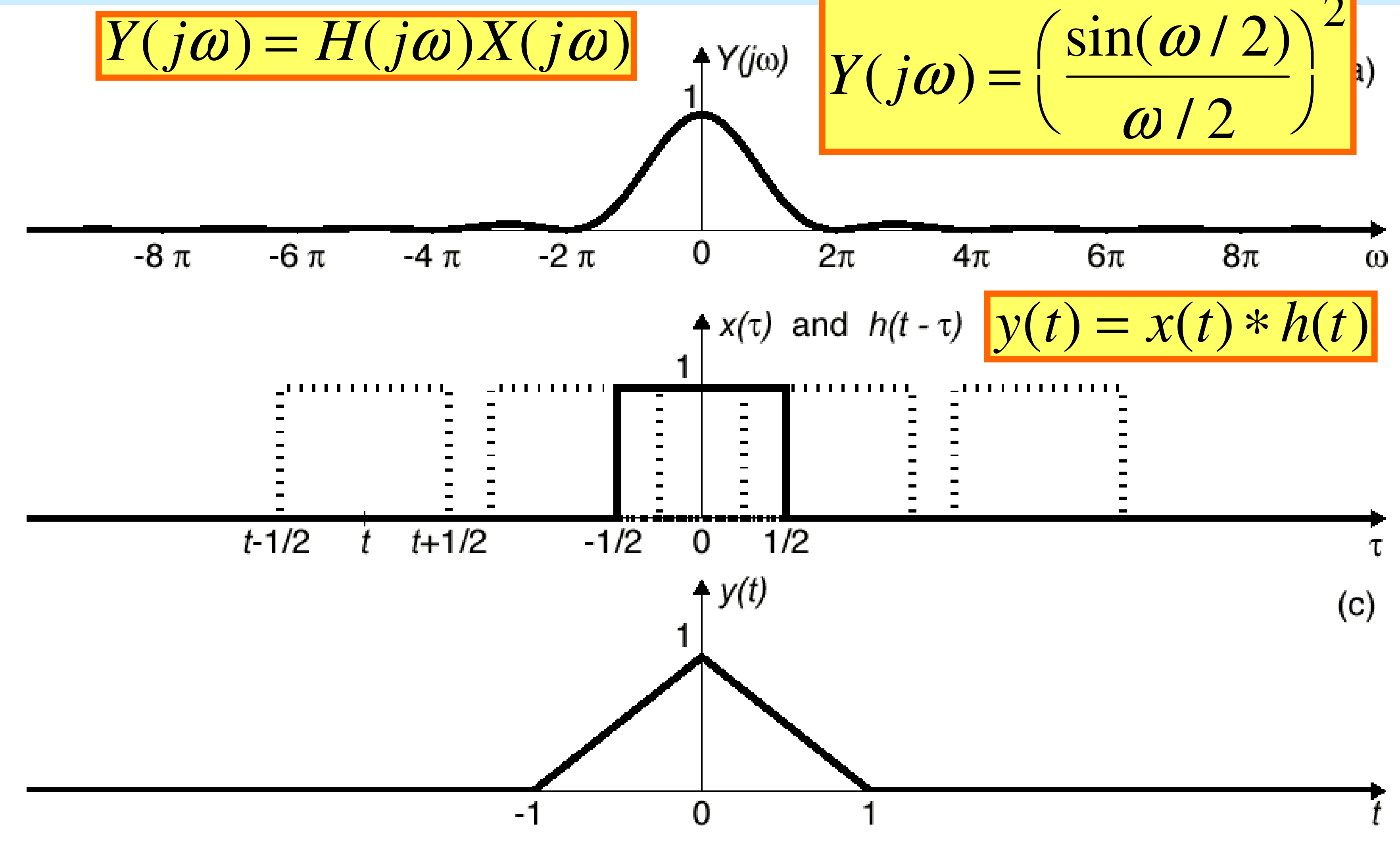


Ex. x(t) and y(t) are both rect. pulse

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$Y(j\omega) = \left(\frac{\sin(\omega/2)}{\omega/2}\right)^2$$

$$y(t) = x(t) * h(t)$$



Cosine Input to LTI System

$$y(t) = h(t) * \cos(\omega_0 t)$$

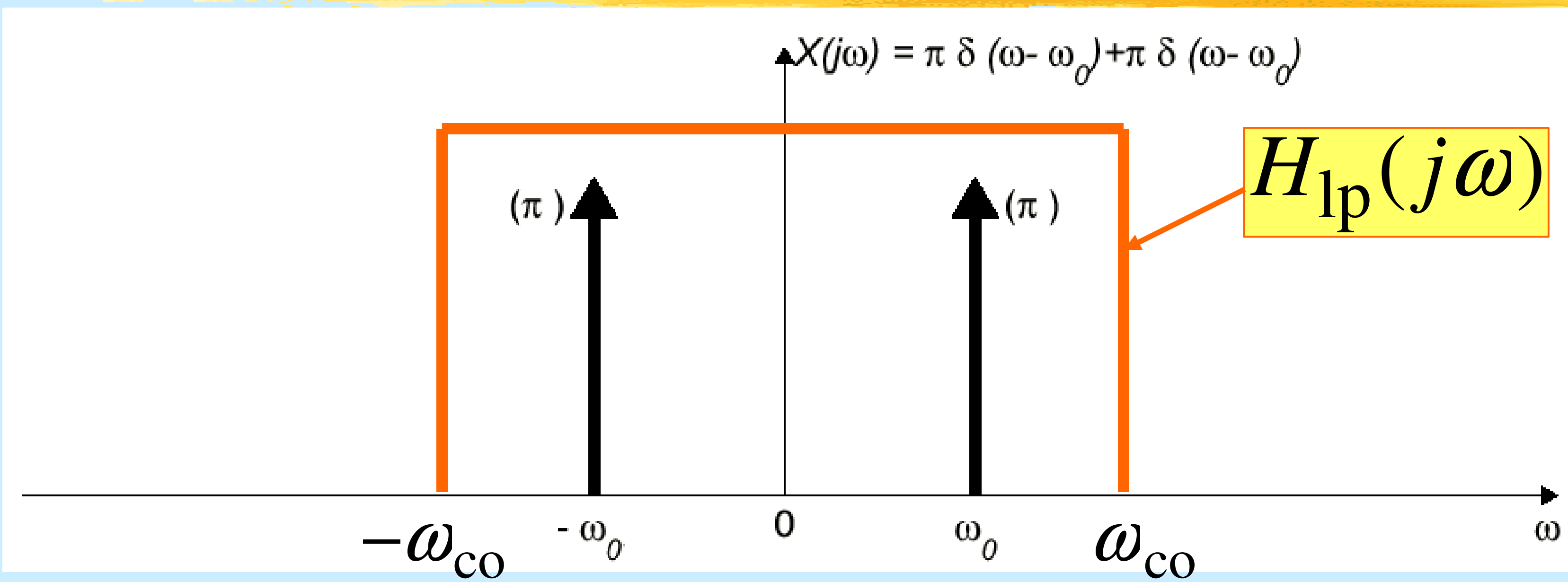
$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$= H(j\omega)[\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)]$$

$$= H(j\omega_0)\pi\delta(\omega - \omega_0) + H(-j\omega_0)\pi\delta(\omega + \omega_0)$$

$$\begin{aligned} y(t) &= H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t} + H(-j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t} \\ &= H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t} + H^*(j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t} \\ &= \left|H(j\omega_0)\right|\cos(\omega_0 t + \angle H(j\omega_0)) \end{aligned}$$

Ideal Lowpass Filter



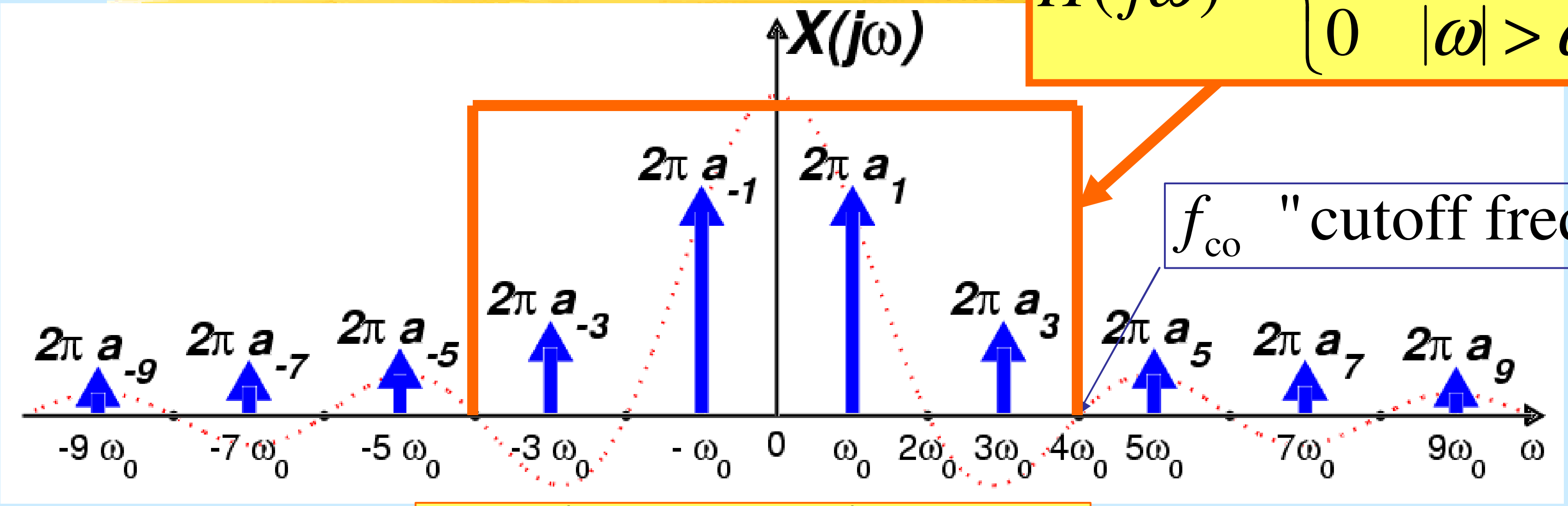
$$y(t) = x(t) \qquad \text{if } \omega_0 < \omega_{\text{co}}$$

$$y(t) = 0 \qquad \text{if } \omega_0 > \omega_{\text{co}}$$

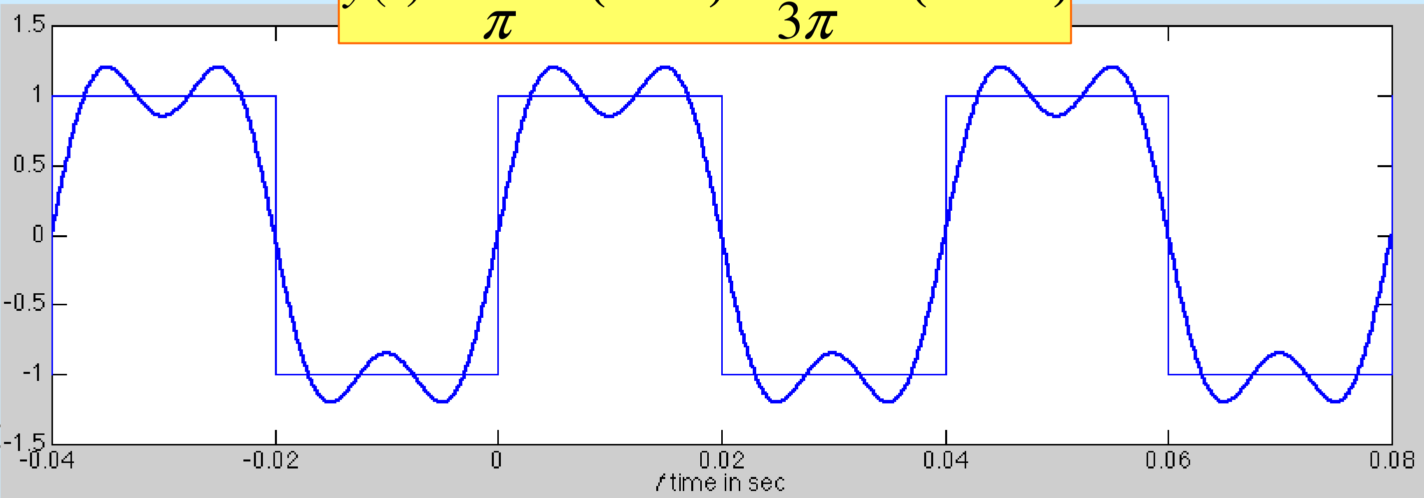
Ideal Lowpass Filter

$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$

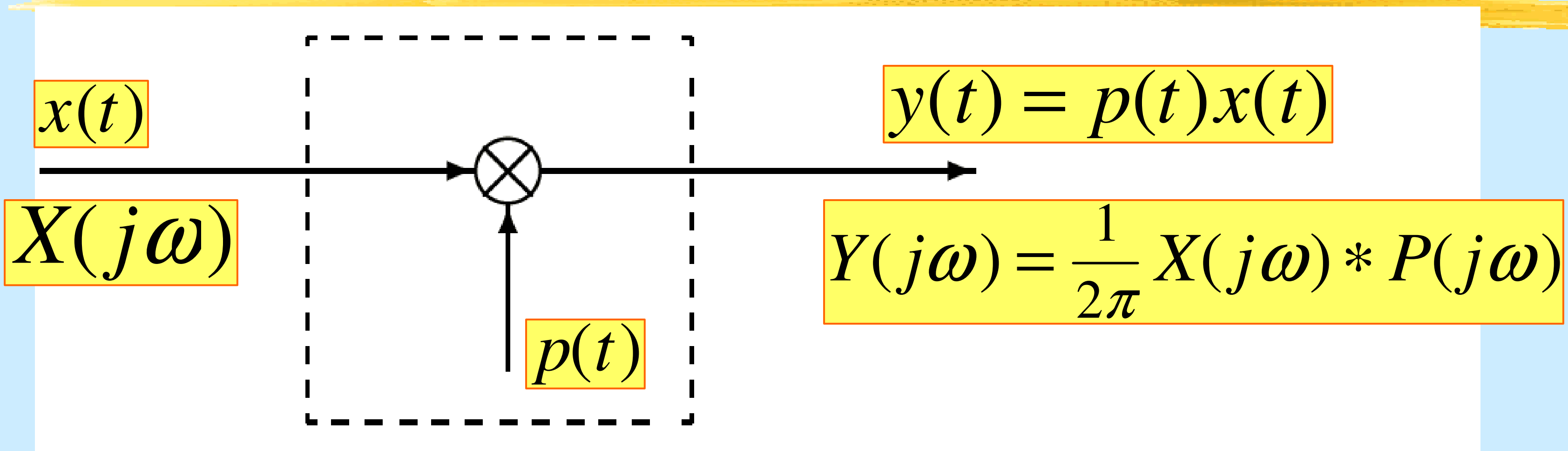
f_{co} "cutoff freq."



$$y(t) = \frac{4}{\pi}\sin(50\pi t) + \frac{4}{3\pi}\sin(150\pi t)$$



Multiplier



- Multiplication in the time-domain corresponds to convolution in the frequency-domain.

$$Y(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta)P(j(\omega-\theta))d\theta$$

Multiply by cos(w0 t)

$$p(t) = \cos(\omega_0 t) \Leftrightarrow P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$y(t) = x(t)p(t) \Leftrightarrow Y(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$y(t) = x(t)\cos(\omega_0 t) \Leftrightarrow$$

$$Y(j\omega) = \frac{1}{2\pi} X(j\omega) * [\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)]$$

$$Y(j\omega) = \frac{1}{2} X(j(\omega - \omega_0)) + \frac{1}{2} X(j(\omega + \omega_0))$$

Differentiation Property

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} (j\omega) X(j\omega) e^{j\omega t} d\omega$$

$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega) X(j\omega)$$

Multiply by $j\omega$

Example

$$e^{-at}u(t) \Leftrightarrow \frac{1}{a+j\omega}$$

$$\frac{d}{dt}\left(e^{-at}u(t)\right) = -ae^{-at}u(t) + e^{-at}\delta(t)$$

$$= \delta(t) - ae^{-at}u(t)$$

$$Y(j\omega) = 1 - \frac{a}{a+j\omega} = \frac{j\omega}{a+j\omega} = j\omega X(j\omega)$$

High order differentiation?

$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega)X(j\omega)$$

$$\frac{d^k x(t)}{dx^k} \Leftrightarrow (j\omega)^k X(j\omega)$$

Proof in class

System of Differential Equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$\Updownarrow$$

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

Recall Difference Equation?

Discrete time system
(Difference equation)

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\Updownarrow$$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Continuous time system
(Differentiation equation)

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$\Updownarrow$$

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

Example systems

- Example systems described by low order differential equations
- How to determine the frequency response
- How to determine the impulse response

Strategy for using the FT

- Develop a set of known Fourier transform pairs.
- Develop a set of “theorems” or properties of the Fourier transform.
- Develop skill in formulating the problem in either the time-domain or the frequency-domain, *which ever leads to the simplest solution*.

Table of Fourier Transforms

$$x(t) = e^{-t}u(t) \quad \Leftrightarrow \quad X(j\omega) = \frac{1}{1 + j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \quad \Leftrightarrow \quad X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$

$$x(t) = \frac{\sin(\omega_b t)}{\pi t} \quad \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}$$

$$x(t) = \delta(t) \quad \Leftrightarrow \quad X(j\omega) = 1$$

$$x(t) = \delta(t - t_0) \quad \Leftrightarrow \quad X(j\omega) = e^{-j\omega t_0}$$

$$x(t) = e^{j\omega_c t} \quad \Leftrightarrow \quad X(j\omega) = 2\pi\delta(\omega - \omega_c)$$

$$x(t) = \cos(\omega_c t) \quad \Leftrightarrow \quad X(j\omega) = \pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c)$$

$$x(t) = \sin(\omega_c t) \quad \Leftrightarrow \quad X(j\omega) = -j\pi\delta(\omega - \omega_c) + j\pi\delta(\omega + \omega_c)$$

Table of Easy FT Properties

Linearity Property

$$a x_1(t) + b x_2(t) \Leftrightarrow a X_1(j\omega) + b X_2(j\omega)$$

Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

Frequency Shifting

$$x(t) e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Scaling

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j(\tfrac{\omega}{a}))$$

Significant FT Properties

$$x(t)*h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

Duality

$$x(t)p(t) \Leftrightarrow \frac{1}{2\pi}X(j\omega)*P(j\omega)$$

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega-\omega_0))$$

$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega)X(j\omega)$$

$$\frac{d^k x(t)}{dx^k} \Leftrightarrow (j\omega)^k X(j\omega)$$

READING ASSIGNMENTS

- This Lecture:
  - Chapter 11, Sects. 11-5 to 11-10
  - **Tables in Section 11-9**
- Other Reading:
  - Entire chap 11