EE3054
Signals and Systems

Sampling of Continuous Time Signals

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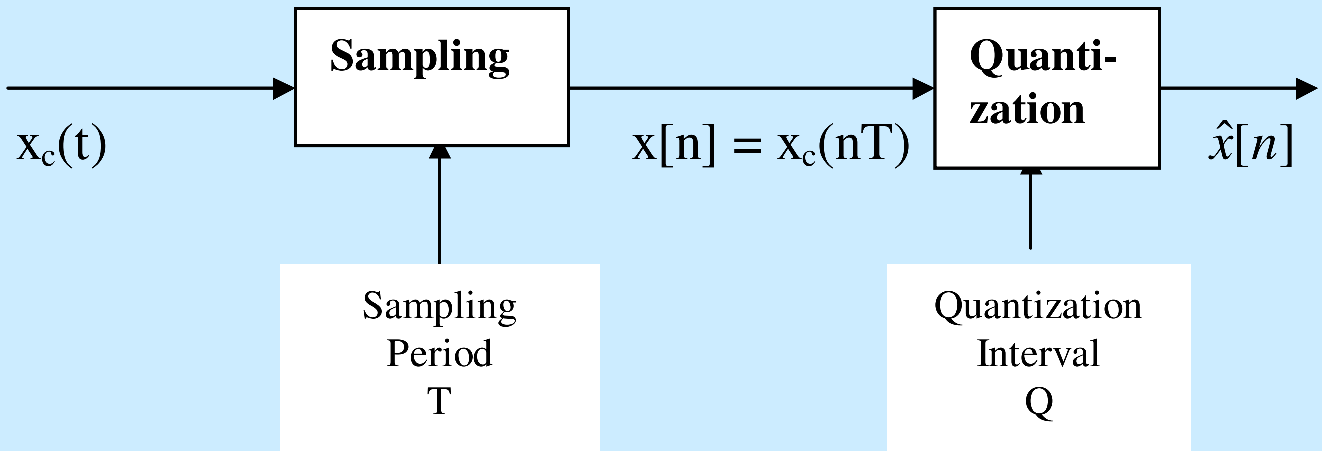
LECTURE OBJECTIVES

- Concept of sampling
- Sampling using periodic impulse train
- Frequency domain analysis
  - Spectrum of sampled signal
  - Nyquist sampling theorem
  - Sampling of sinusoids

Two Processes in A/D Conversion



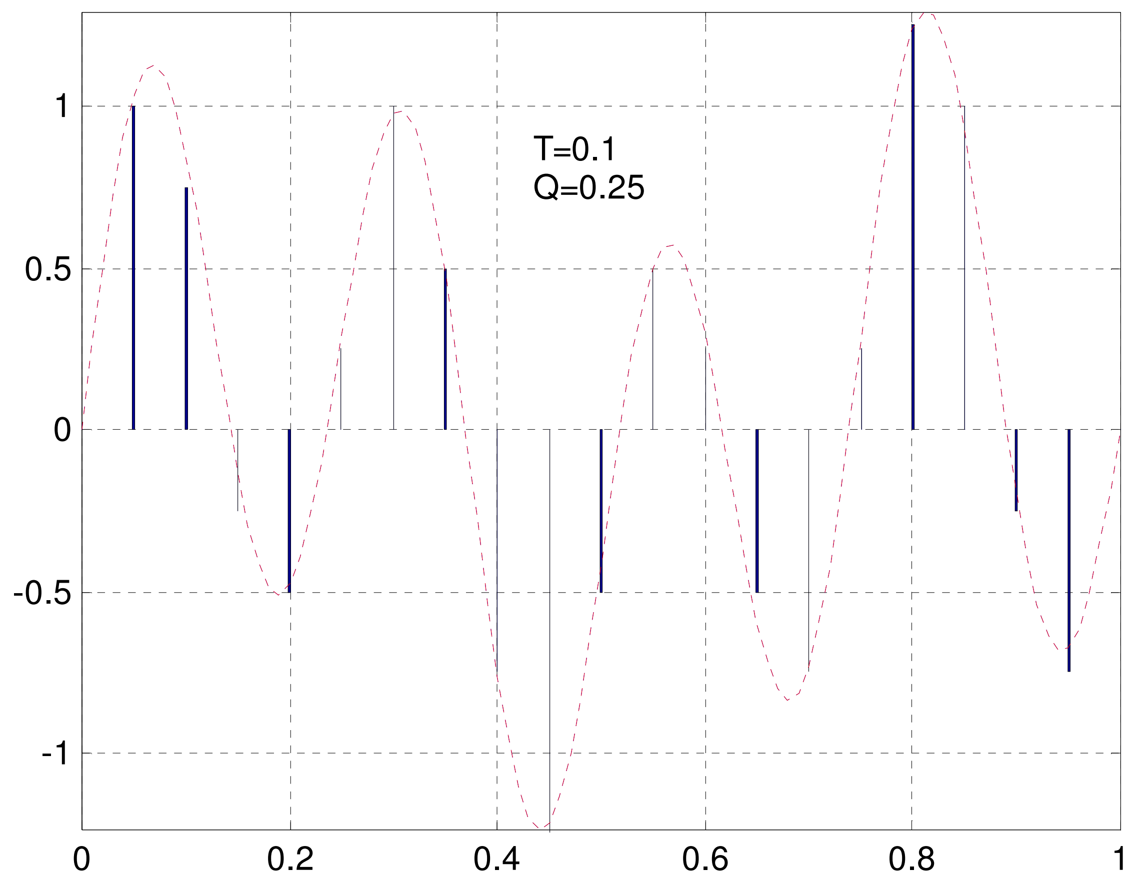
- Sampling: take samples at time *nT*
  - T: sampling period;
  - $f_s = 1/T$: sampling frequency

$$x[n] = x(nT), -\infty < n < \infty$$

- Quantization: map amplitude values into a set of discrete values $\pm pQ$
  - Q: quantization interval or stepsize

$$\hat{x}[n] = Q[x(nT)]$$

Analog to Digital Conversion



A2D_plot.m

How to determine T and Q?

- T (or f_s) depends on the signal frequency range
  - A fast varying signal should be sampled more frequently!
  - Theoretically governed by the Nyquist sampling theorem
    - $f_s > 2 f_m$ (f_m is the maximum signal frequency)
    - For speech: $f_s >= 8$ *KHz;* For music: $f_s >= 44$ *KHz;*
- Q depends on the dynamic range of the signal amplitude and perceptual sensitivity
  - Q and the signal range D determine bits/sample R
    - $2^R = D/Q$
    - For speech: $R = 8$ *bits;* For music: $R = 16$ *bits;*
- One can trade off T (or f_s) and Q (or R)
  - lower R -> higher f_s; higher R -> lower f_s
- We only consider sampling in this class

SAMPLING x(t)

- SAMPLING PROCESS
  - Convert x(t) to numbers x[n]
  - "n" is an integer; x[n] is a sequence of values
  - Think of "n" as the storage address in memory
- UNIFORM SAMPLING at $t = nT_s$
  - IDEAL: $x[n] = x(nT_s)$

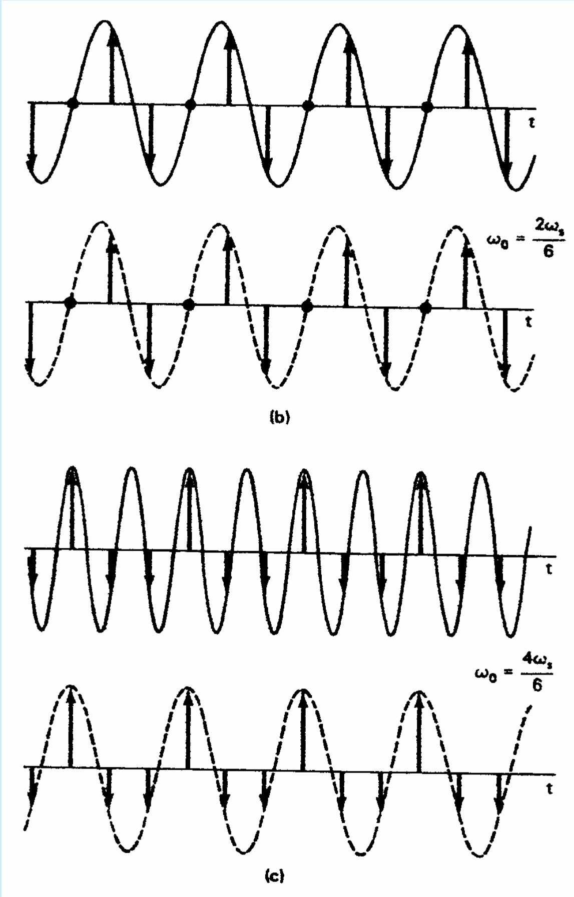
Sampling of Sinusoid Signals

Sampling above Nyquist rate
$\omega_s=3\omega_m>\omega_{s0}$

Reconstructed =original

Sampling under Nyquist rate
$\omega_s=1.5\omega_m<\omega_{s0}$

Reconstructed \= original

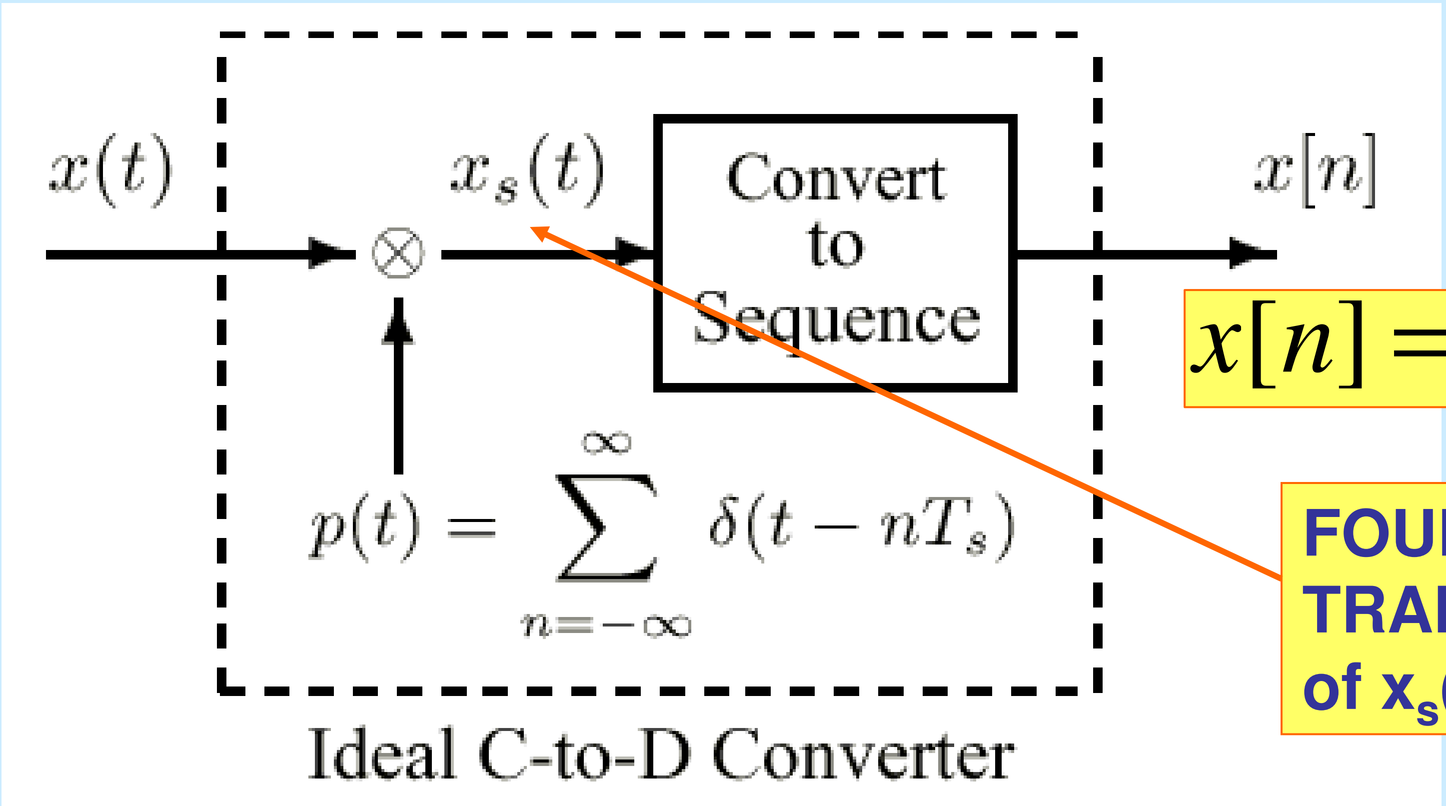


Aliasing: The reconstructed sinusoid has a lower frequency than the original!

Nyquist Sampling Theorem

- Theorem:
  - If x(t) is bandlimited, with maximum frequency f_b(or $\omega_b = 2\pi f_b$)
  - and if $f_s = 1/T_s > 2 f_b$ or $\omega_s = 2\pi / T_s > 2 \omega_b$
  - Then $x_c(t)$ can be reconstructed perfectly from x[n]= x(nT_s) by using an ideal low-pass filter, with cut-off frequency at $f_s/2$
  - $f_{s0} = 2 f_b$ is called the *Nyquist Sampling Rate*
- Physical interpretation:
  - Must have at least two samples within each cycle!

Sampling Using Periodic Impulse Train



$$x[n] = x(nT_s)$$

FOURIER TRANSFORM of $x_s(t)$???

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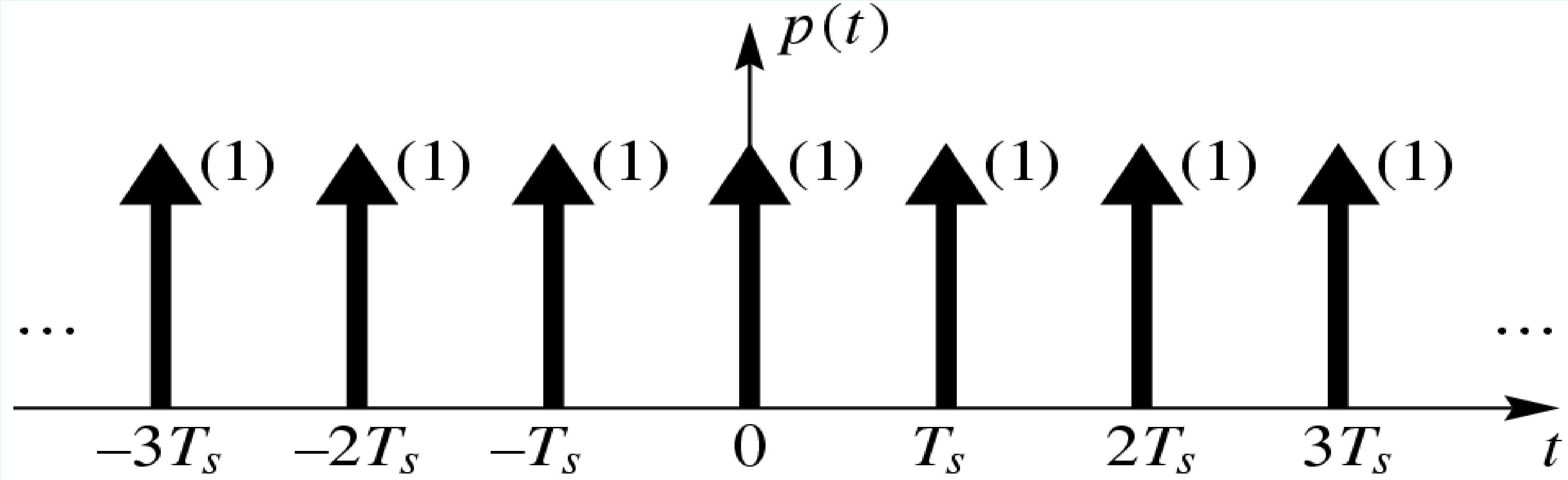
Periodic Impulse Train



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

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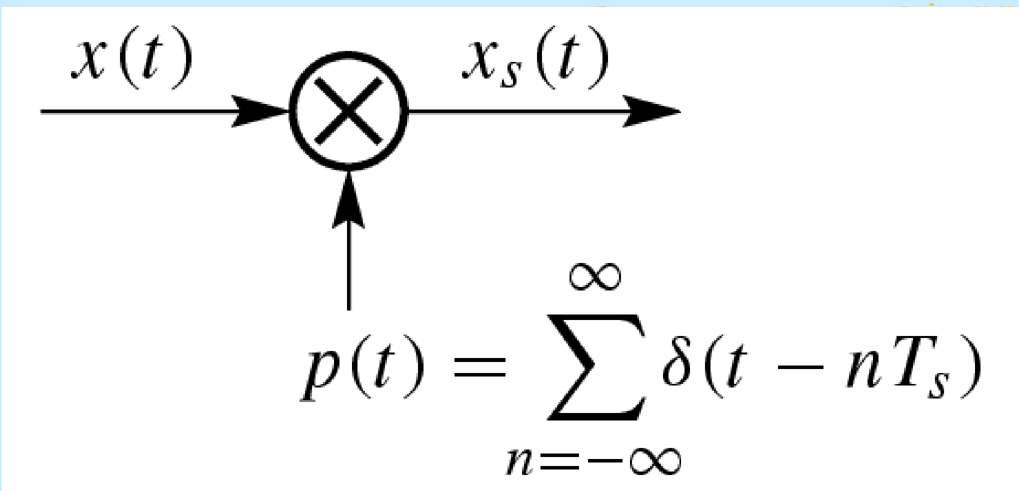
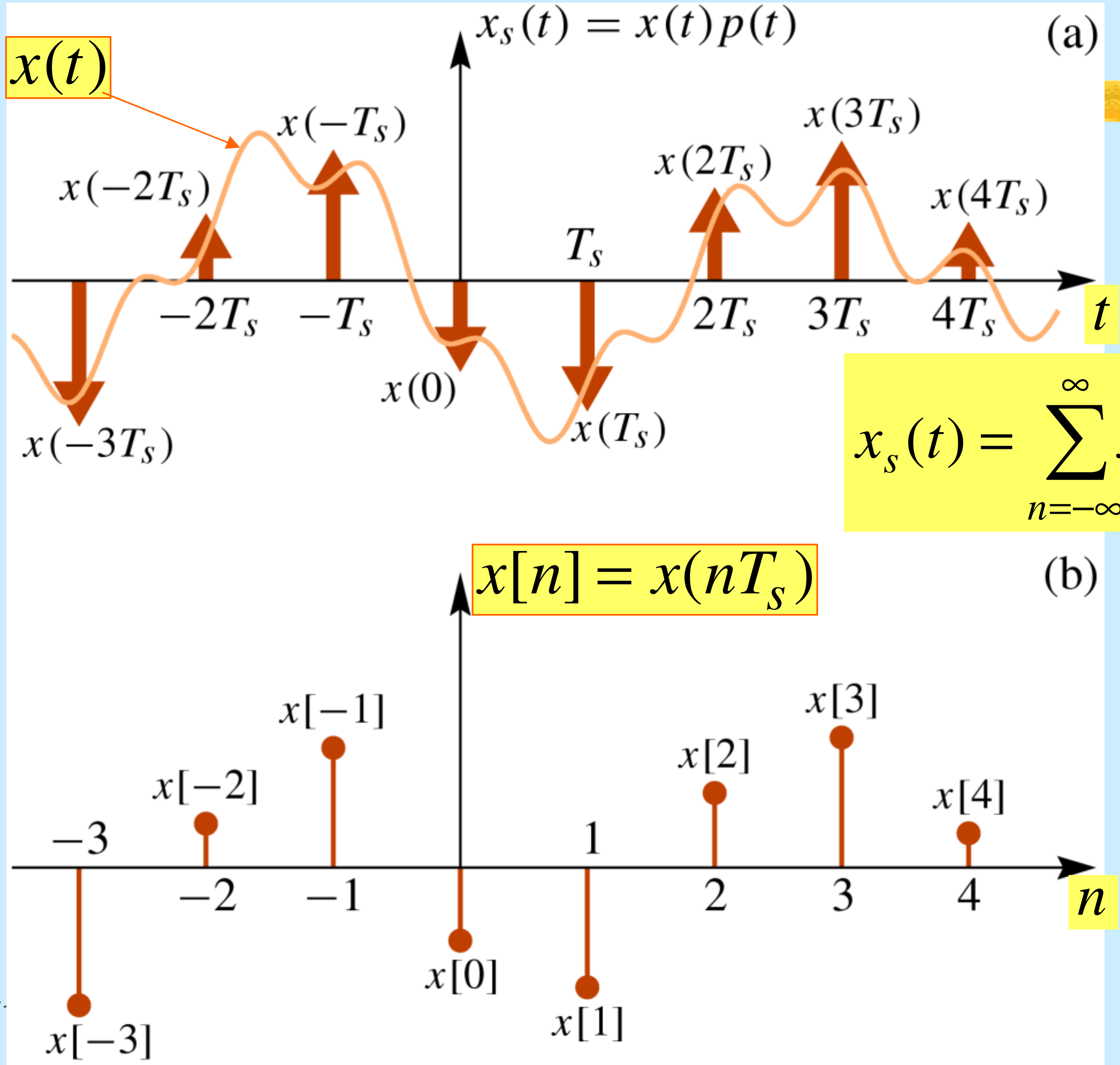
Impulse Train Sampling



$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} \underline{x(t)\delta(t - nT_s)}$$

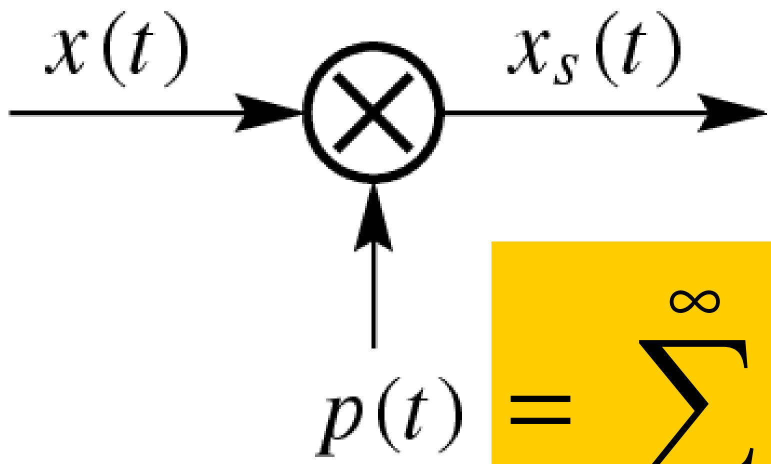
$$\boxed{x_s(t) = \sum_{n=-\infty}^{\infty} \underline{x(nT_s)\delta(t - nT_s)}}$$

Illustration of Sampling



$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s)$$

Sampling: Freq. Domain



$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

How is the spectrum of $x_s(t)$ related to that of $x(t)$?

EXPECT FREQUENCY SHIFTING !!!

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

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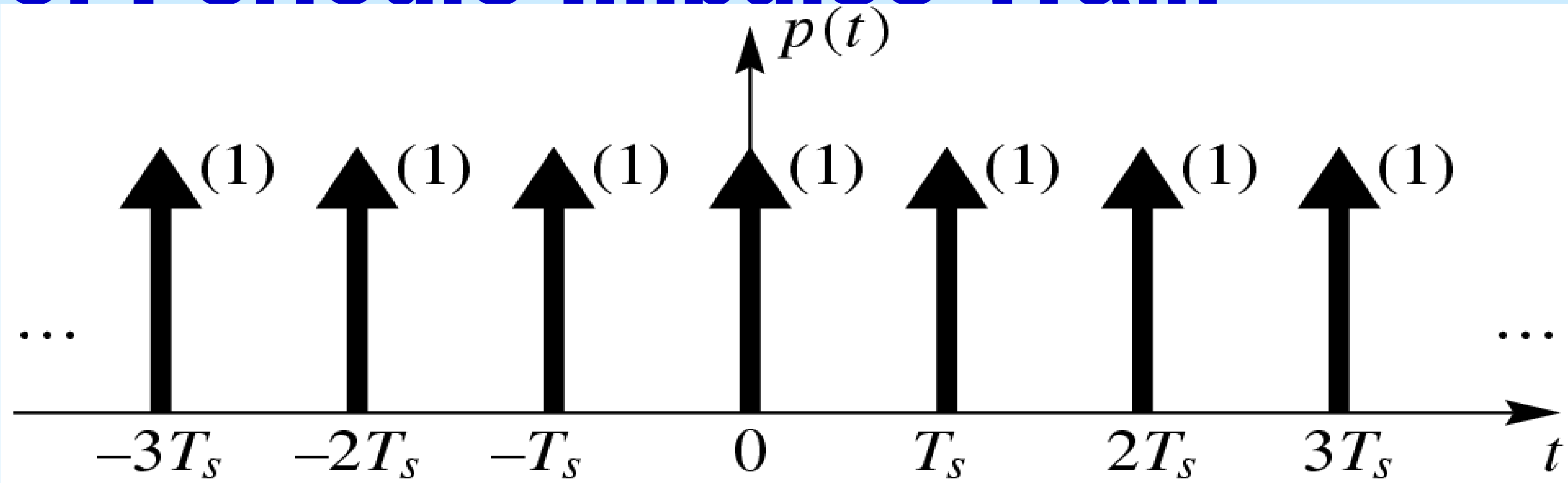
Fourier Series Representation of Periodic Impulse Train



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

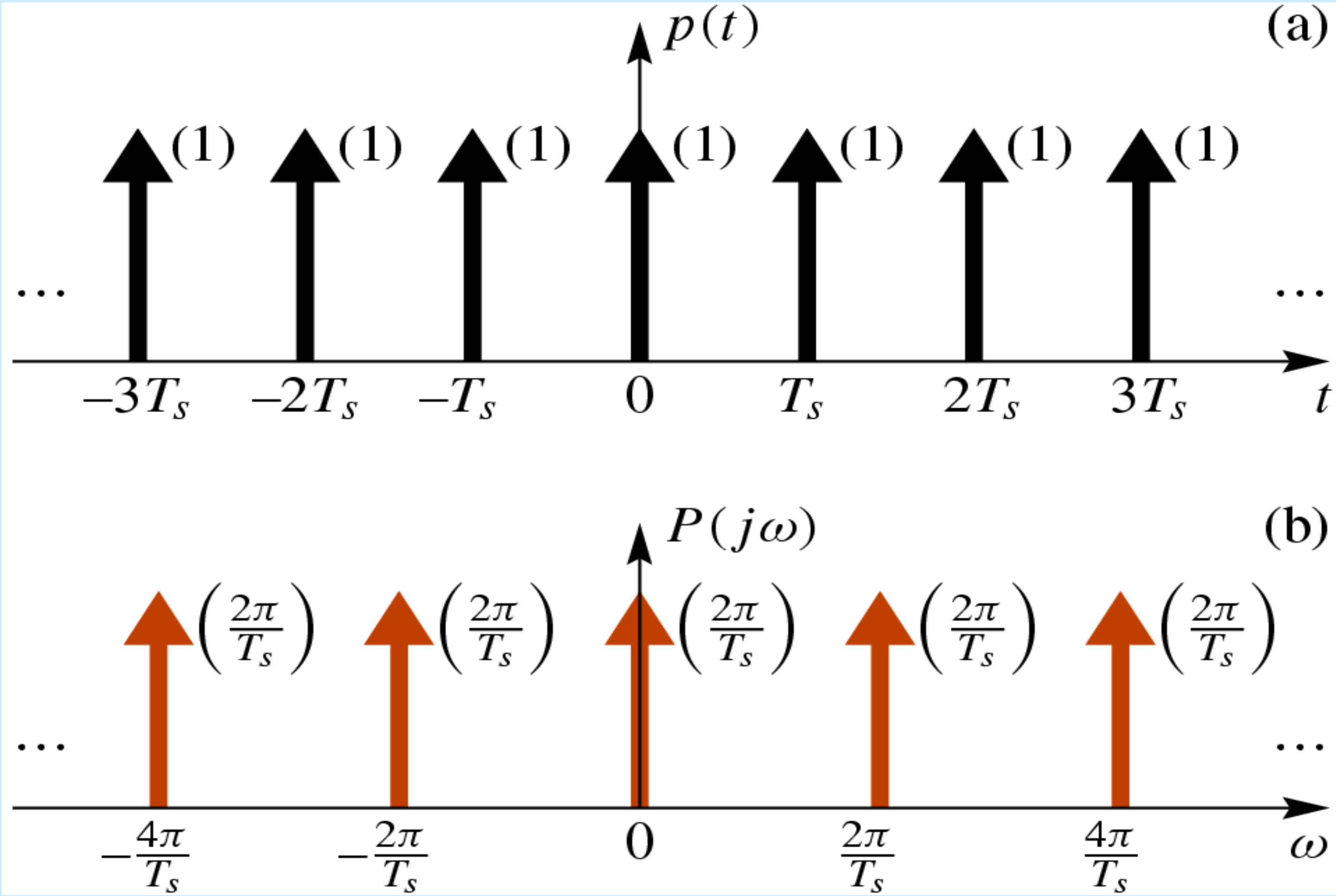
$$\omega_s = \frac{2\pi}{T_s}$$

$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T_s}$$

Fourier Series

FT of Impulse Train

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \frac{1}{T_s} \sum_{k} e^{jk\omega_s t} \quad \leftrightarrow \quad P(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T_s} \delta(\omega - k\omega_s)$$



$$\omega_s = \frac{2\pi}{T_s}$$

Frequency-Domain Analysis: Using Fourier Series

$$x_s(t) = x(t)\,p(t)$$

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \frac{1}{T_s} \sum_{k} e^{jk\omega_s t}$$

$$x_s(t) = x(t) \sum_{k=-\infty}^{\infty} \frac{1}{T_s} e^{jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \underline{x(t) e^{jk\omega_s t}}$$

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \underline{X(j(\omega - k\omega_s))}$$

$$\omega_s = \frac{2\pi}{T_s}$$

Frequency-Domain Analysis: Using Multiplication-Convolution duality

$$p(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT_s)=\frac{1}{T_s}\sum_{k}e^{jk\omega_s t} \quad \leftrightarrow \quad P(j\omega)=\sum_{k=-\infty}^{\infty}\frac{2\pi}{T_s}\delta(\omega-k\omega_s)$$

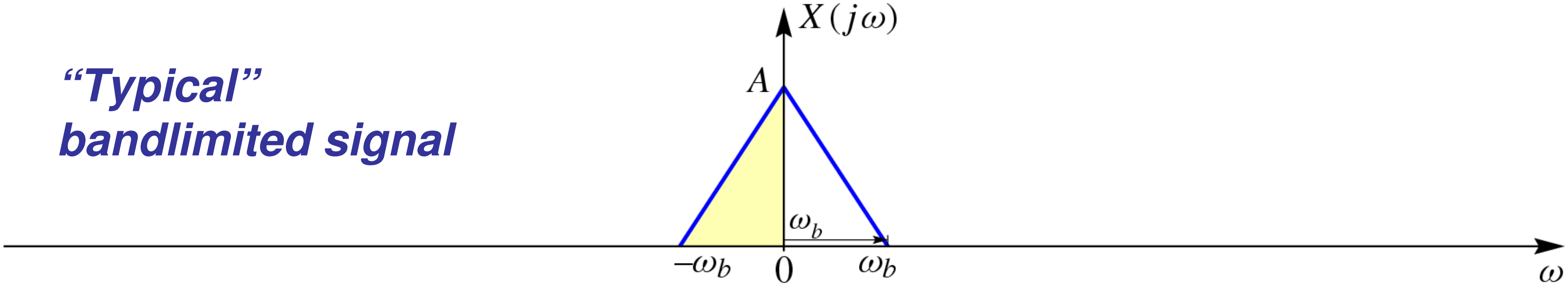
$$\boxed{x(t)p(t)\Leftrightarrow\frac{1}{2\pi}X(j\omega)*P(j\omega)}$$

$$X_s(j\omega)=\frac{1}{2\pi}X(j\omega)*P(j\omega)=\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\frac{2\pi}{T_s}X(j\omega)*\delta(\omega-k\omega_s)$$

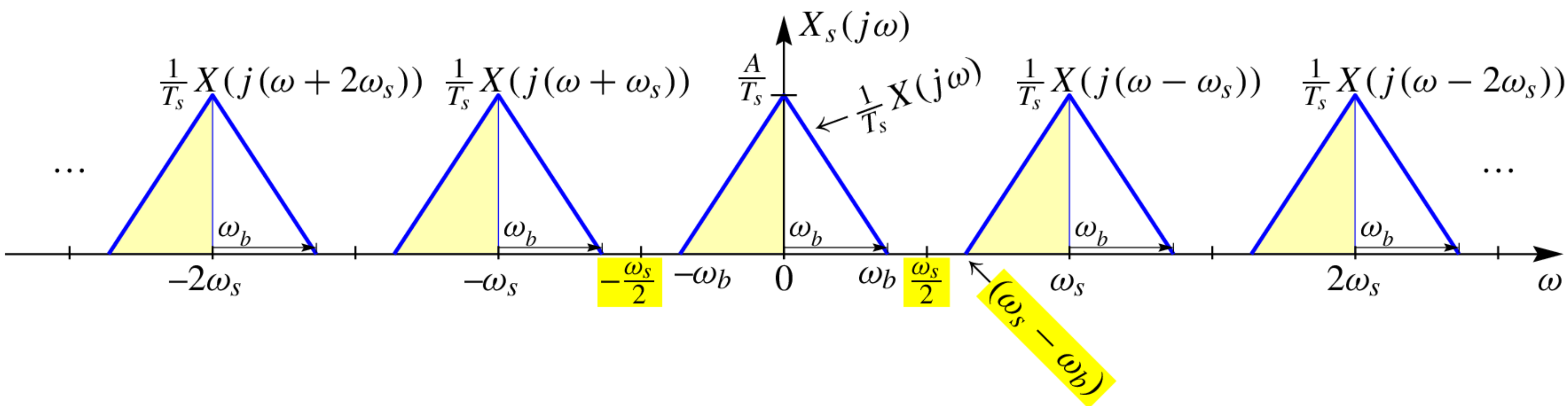
$$=\frac{1}{T_s}\sum_{k=-\infty}^{\infty}X\left(j(\omega-k\omega_s)\right)$$

Frequency-Domain Representation of Sampling

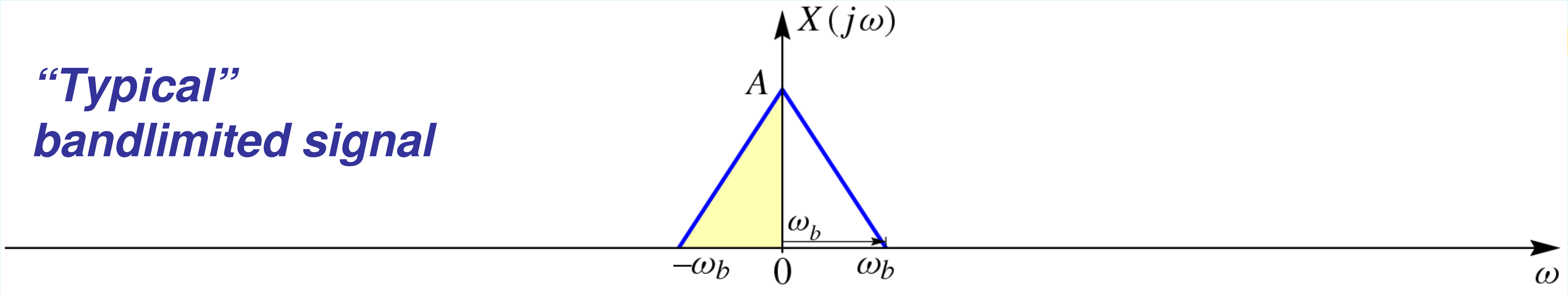
"Typical" bandlimited signal



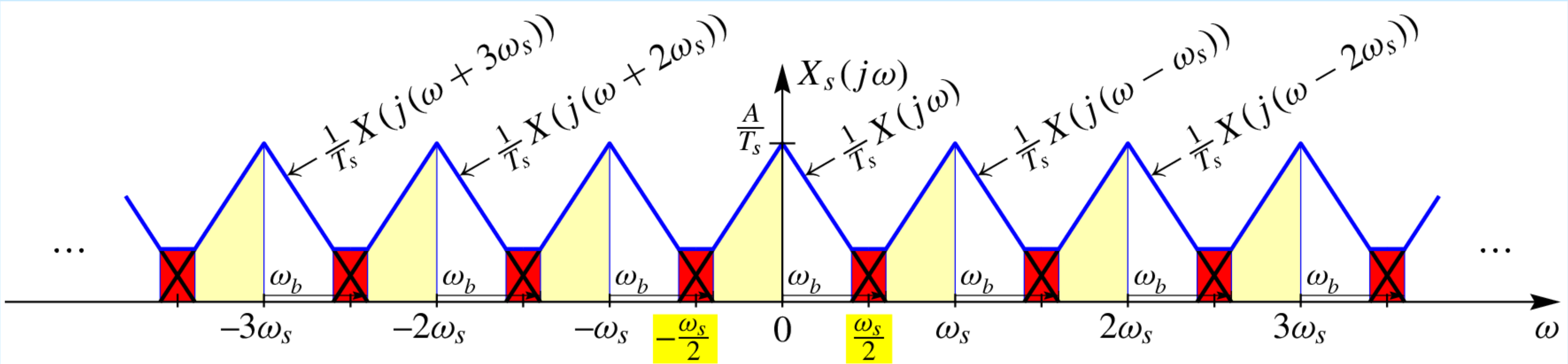
$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$



Aliasing Distortion



- If $\omega_s < 2\omega_b$, the copies of $X(j\omega)$ overlap, and we have aliasing distortion.



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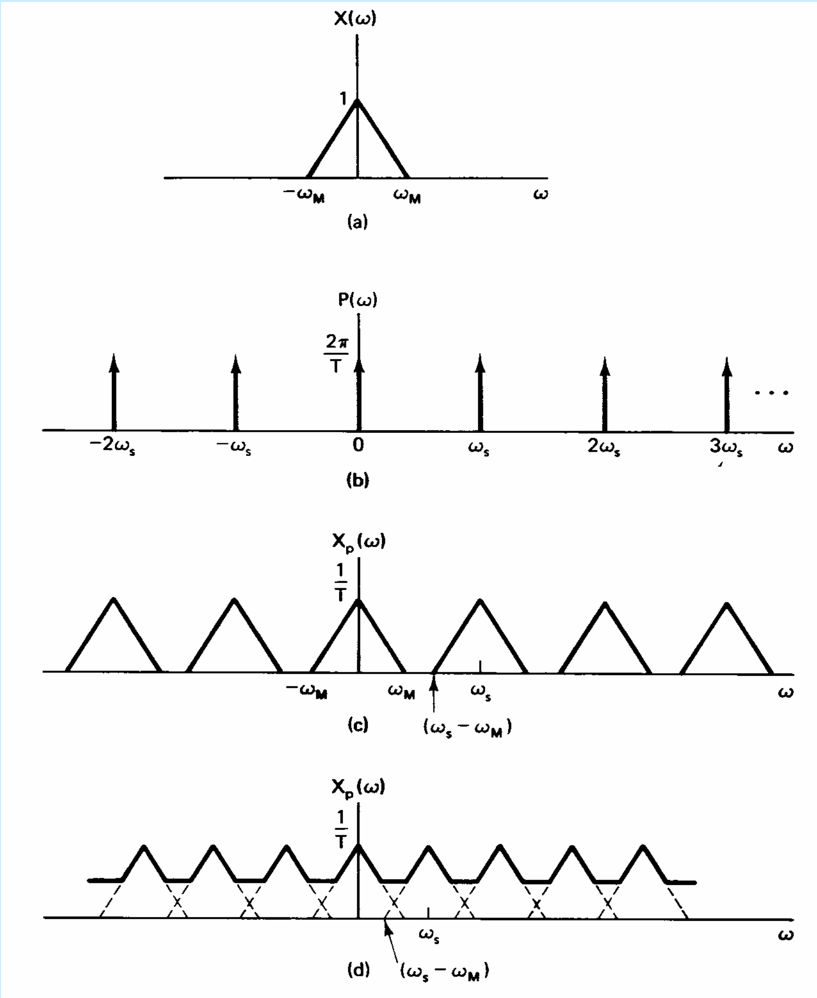
Frequency Domain Interpretation of Sampling

Original signal

Sampling impulse train

Sampled signal $\omega_s > 2\,\omega_m$

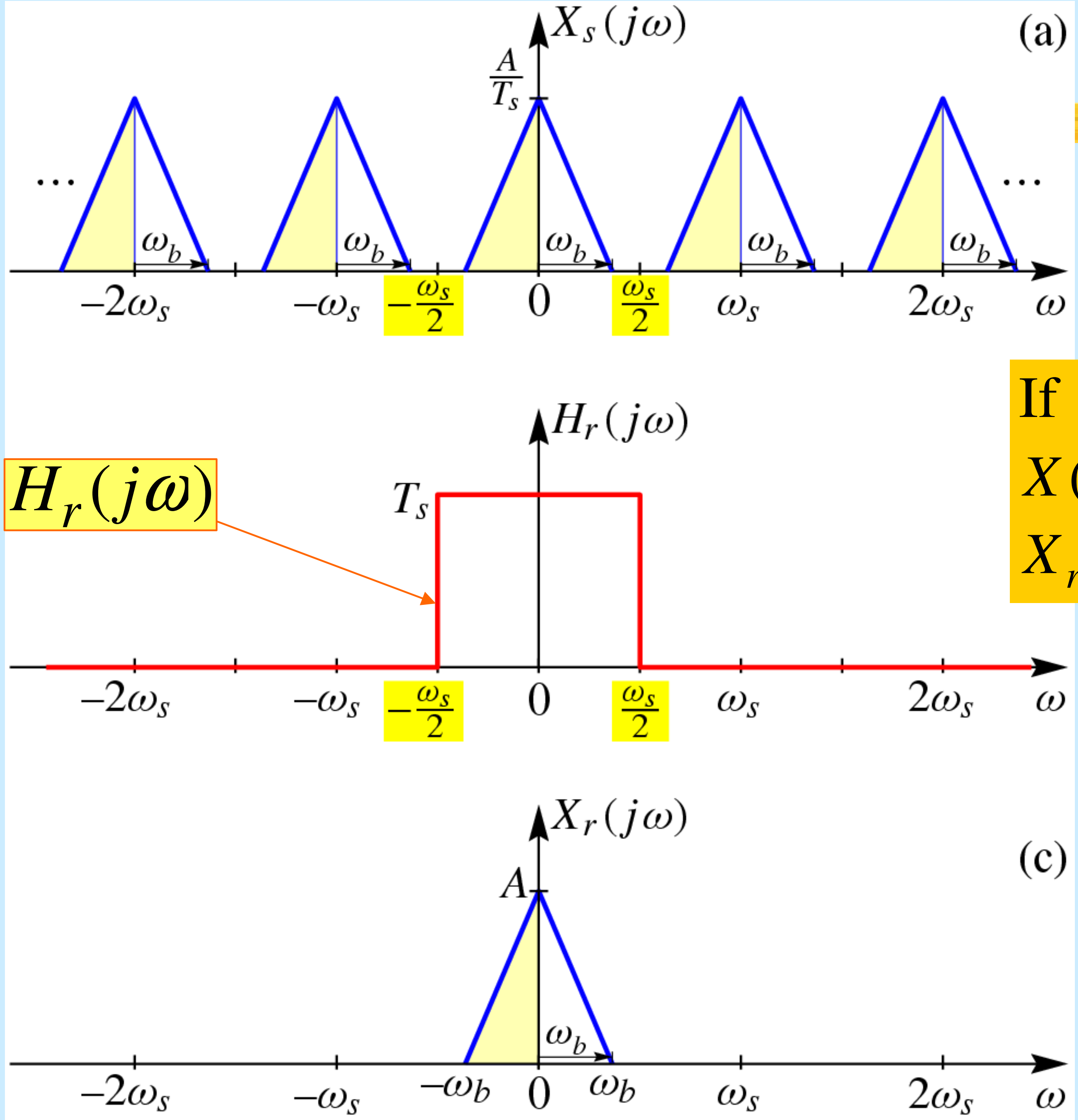
Sampled signal $\omega_s < 2\,\omega_m$ (Aliasing effect)



The spectrum of the sampled signal includes the original spectrum and its aliases (copies) shifted to *k* f_s, *k=+/- 1,2,3,…*
The reconstructed signal from samples has the frequency components upto $f_s/2$.

When $f_s < 2f_m$*, aliasing occur.*

Reconstruction: Frequency-Domain



If $\omega_s > 2\omega_b$, the copies of $X(j\omega)$ do not overlap, so $X_r(j\omega) = H_r(j\omega)X_s(j\omega)$

Nyquist Sampling Theorem

- Theorem:
  - If x(t) is bandlimited, with maximum frequency f_b (or $\omega_b = 2\pi f_b$)
  - and if $f_s = 1/T_s > 2 f_b$ or $\omega_s = 2\pi / T_s > 2\omega_b$
  - Then $x_c(t)$ can be reconstructed perfectly from x[n]= x(nT_s) by using an ideal low-pass filter, with cut-off frequency at $f_s/2$
  - $f_{s0} = 2 f_b$ is called the *Nyquist Sampling Rate*

- Physical interpretation:
  - Must have at least two samples within each cycle!

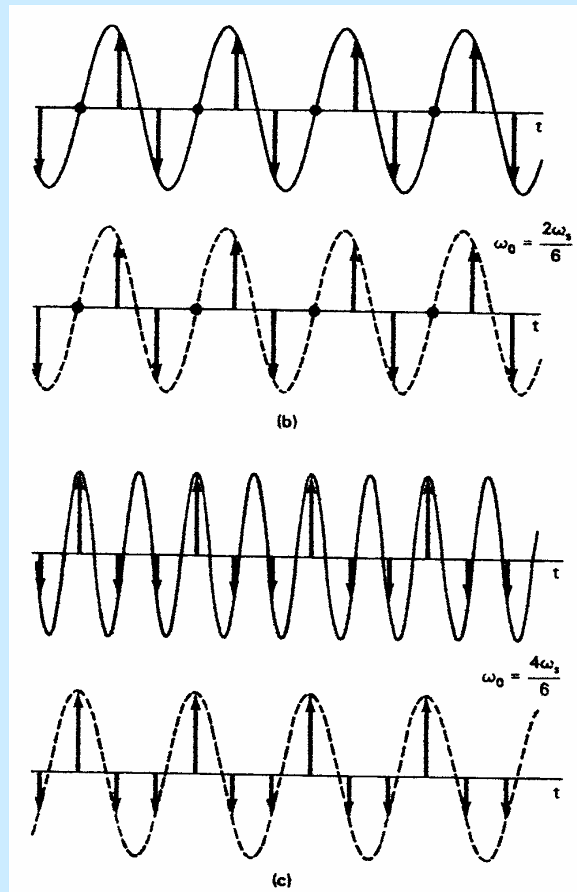
Sampling of Sinusoid Signals: Temporal domain

Sampling above Nyquist rate
$\omega_s=3\omega_m>\omega_{s0}$

Reconstructed
=original

Sampling under Nyquist rate
$\omega_s=1.5\omega_m<\omega_{s0}$

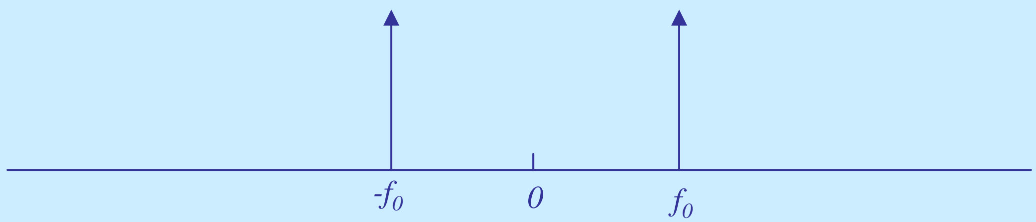
Reconstructed
\= original



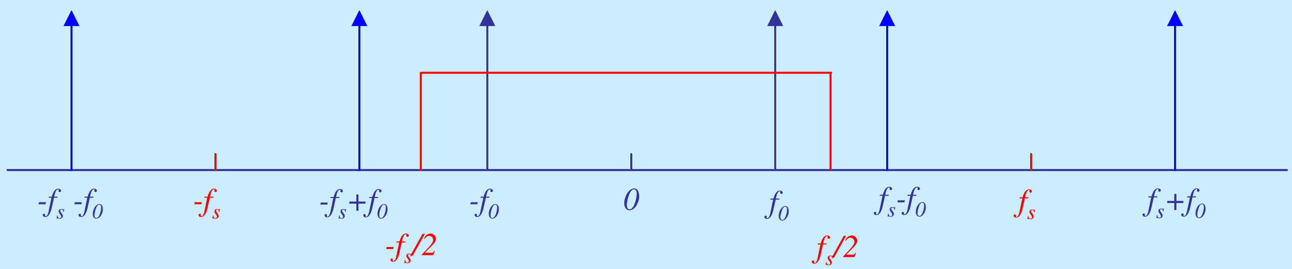
Aliasing: The reconstructed sinusoid has a lower frequency than the original!

Sampling of Sinusoid: Frequency Domain

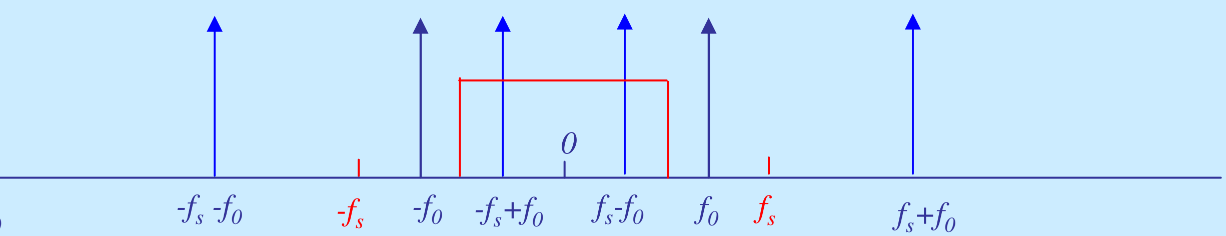
Spectrum of
$cos(2\pi f_0 t)$



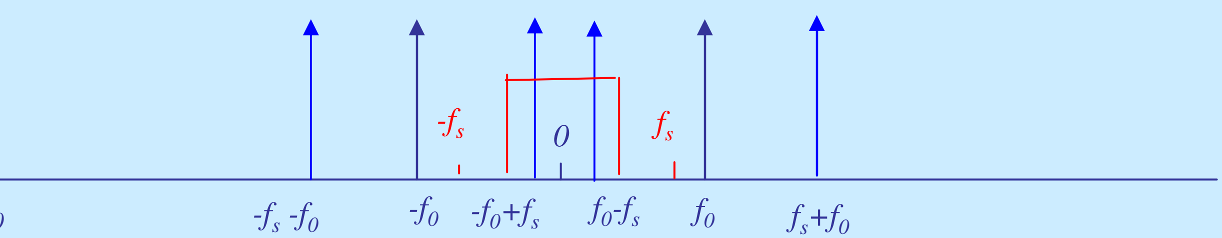
No aliasing
$f_s > 2f_0$
$f_s - f_0 > f_0$
Reconstructed signal: f_0



With aliasing
$f_0 < f_s < 2f_0$ (folding)
$f_s - f_0 < f_0$
Reconstructed signal: $f_s - f_0$

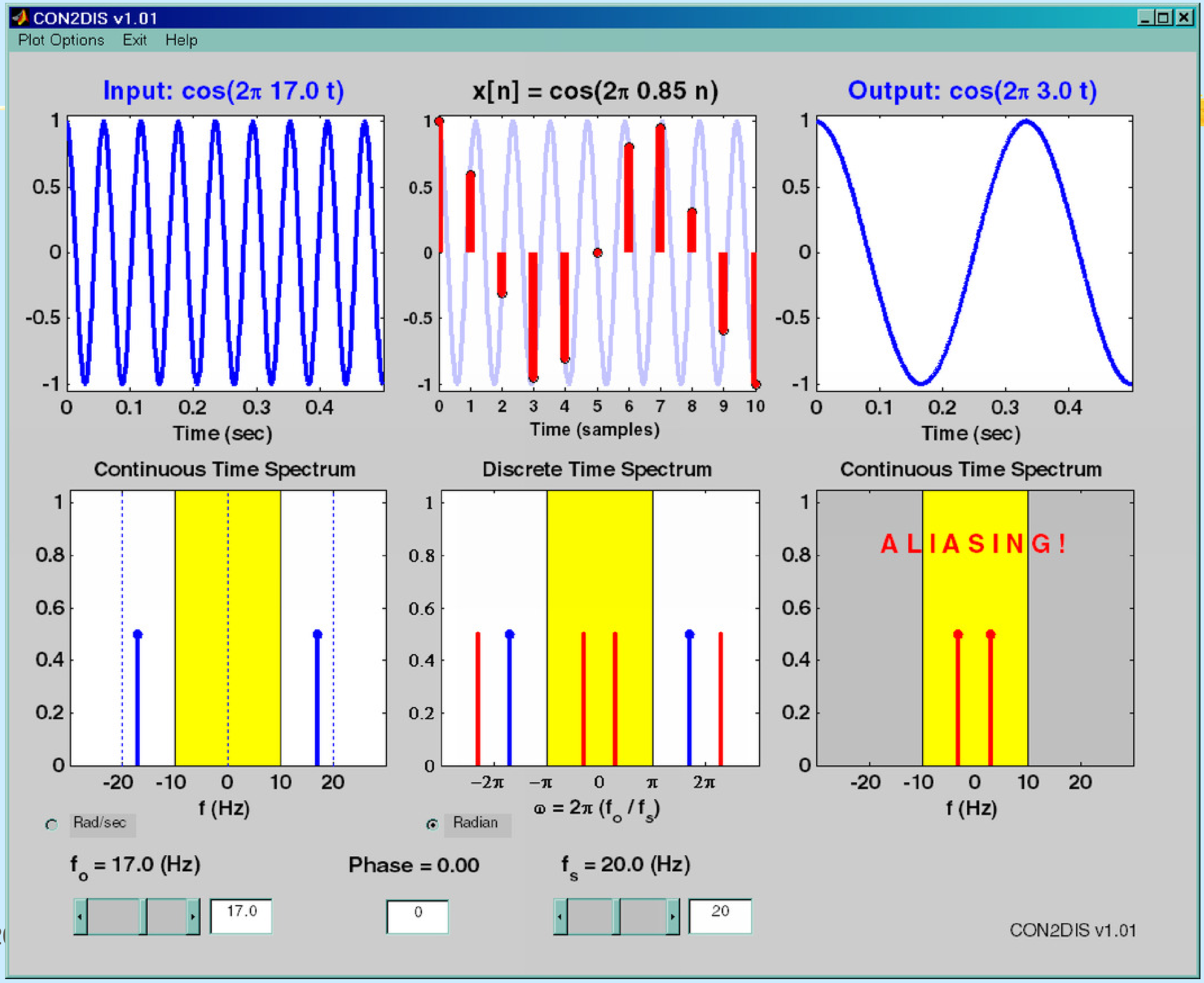


With aliasing
$f_s < f_0$ (aliasing)
$f_0 - f_s < f_0$
Reconstructed signal: $f_s - f_0$



More examples with Sinusoids

SAMPLING GUI (con2dis)

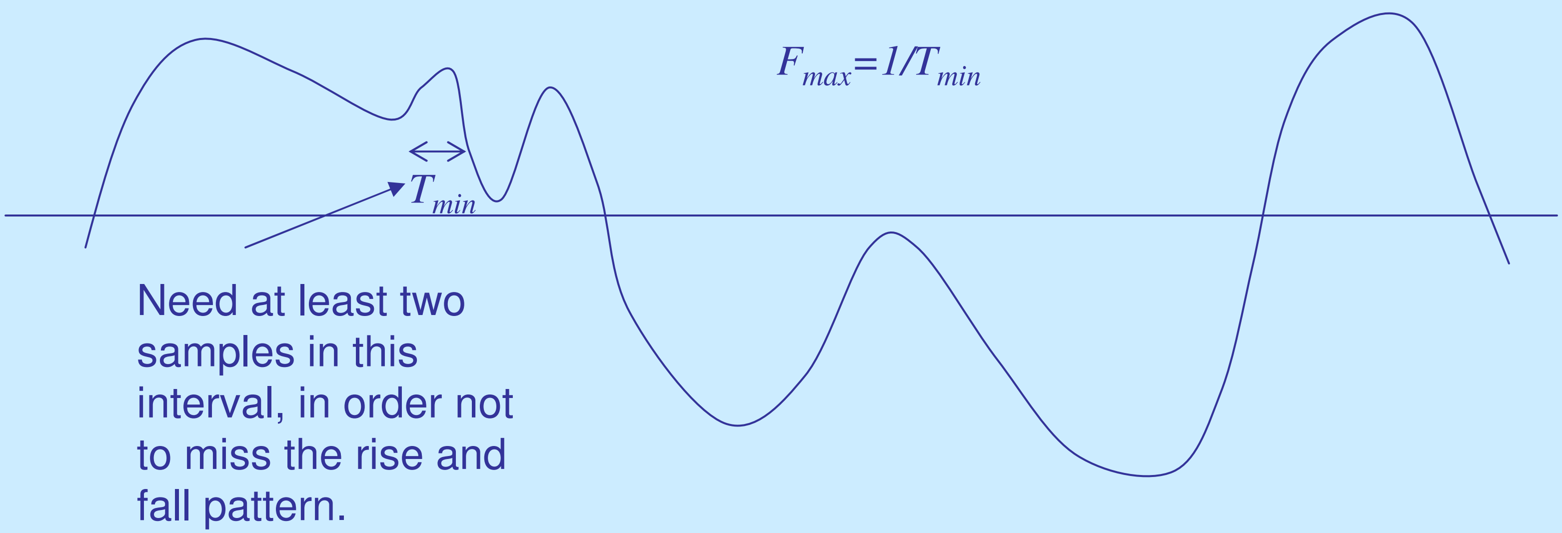


Strobe Movie

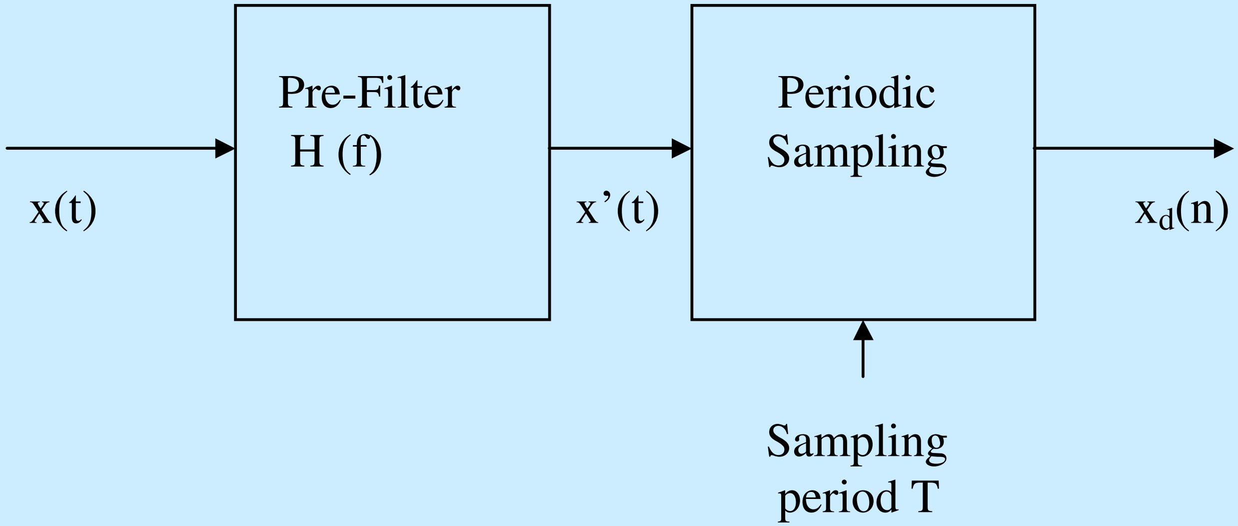
- From SP First, Chapter 4, Demo on "Strobe Movie"

How to determine the necessary sampling frequency from a signal waveform?

- Given the waveform, find the shortest ripple, there should be at least two samples in the shortest ripple
- The inverse of its length is approximately the highest frequency of the signal



Sampling with Pre-Filtering



- If $f_s < 2f_b$, aliasing will occur in sampled signal
- To prevent aliasing, pre-filter the continuous signal so that $f_b<f_s/2$
- Ideal filter is a low-pass filter with cutoff frequency at $f_s/2$ (corresponding to sync functions in time)
- Common practical pre-filter: averaging within one sampling interval

Summary

- Sampling as multiplication with the periodic impulse train
- FT of sampled signal: original spectrum plus shifted versions (aliases) at multiples of sampling freq.
- Sampling theorem and Nyquist sampling rate
- Sampling of sinusoid signals
  - Can illustrate what is happening in both temporal and freq. domain. Can determine the reconstructed signal from the sampled signal.
- Need for prefilter
- Next lecture: how to recover continuous signal from samples, ideal and practical approaches

Readings

- Textbook: Sec. 12.3.1-12.3.2, 4.1-4.3
- Oppenheim and Willsky, *Signals and Systems*, Chap. 7.
  - Optional reading (More depth in frequency domain interpretation)