THE SLANTLET TRANSFORM

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ABSTRACT

The discrete wavelet transform (DWT) is usually carried out by filter bank iteration, however, for a fixed number of zero moments, this does not yield a discrete-time basis that is optimal with respect to time-localization. This paper discusses the implementation and properties of an orthogonal DWT, with two zero moments and with improved time-localization. The basis, is not based on filter bank iteration, instead different filters are used for each scale. For coarse scales, the support of the discrete-time basis functions approaches 2/3 that of the corresponding functions obtained by filter bank iteration. This slantlet basis is piecewise linear and retains the octave-band characteristic. Closed form expressions for the filters are given and improvement in a denoising example is shown. This bases, being piecewise linear, is reminiscent of the slant transform, to which it is compared.

1. INTRODUCTION

Discrete wavelet transforms (DWTs) are useful in a variety of applications (such as estimation, compression, and fast algorithms) partly because they can provide relatively efficient representations of piecewise smooth signals. The degree to which a wavelet basis can successfully yield sparse representations of such signals depends on the time-localization and smoothness properties of the basis functions. For example, signal smoothing by the nonlinear thresholding of DWT coefficients [6] preserves edges reasonably well — in part, because the support of each basis function is short with respect to its bandwidth. (Thus the "constant-Q" behavior, or octave-band characteristic, of the associated filter bank.) However, a fundamental trade-off exists between time-localization and smoothness characteristics, and it is desirable to obtain a good trade-off between these two competing criteria in designing wavelet bases. As is usual, in this paper, the lengths of the discrete-time basis functions and their moments are the vehicles by which good time-localization and smoothness properties are achieved.

Although the usual filter bank iteration provides a simple way to generate an orthogonal discrete-time basis having an octave-band characteristic, for a fixed number of zero moments it does not yield a discrete-time basis that is optimal with respect to time-localization. This paper examines a special case of a class of bases originally described by Alpert in [3, 4] in a multiwavelet context, the construction of which relies on Gram-Schmidt orthogonalization. We describe the basis from a filter bank viewpoint and give explicit solutions for the filter coefficients. The DWT described in this paper is based on a filter bank structure, where different filters are used for each scale — however a very simple efficient algorithm based on recursion is available, although not given here due to space limitations. For the DWT filter bank described here, the support of the discrete-time basis functions is reduced (by a factor approaching 1/3 for coarse scales), while retaining the basic characteristics of the 2-band iterated filter bank tree. This basis retains the octave-band characteristic We wish to note that [2] also describes a transform that changes from scale to scale.

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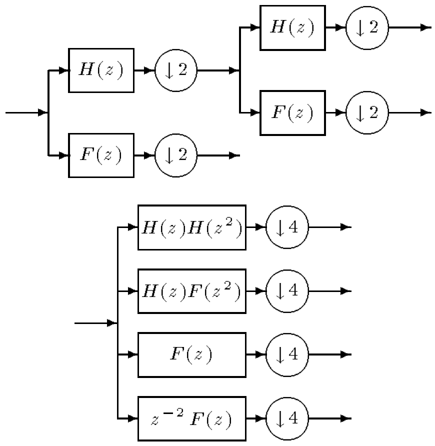

Figure 1: Iterated filter bank and an equivalent structure.

2. SLANTLET FILTER BANK

It is useful to consider first the usual iterated DWT filter bank and an equivalent form, shown in Figure 1. The 'slantlet' filter bank described here is based on the second structure, but it will be occupied by different filters, that are not products. With the extra degrees of freedom obtained by giving up the product form, it is possible to design filters of shorter length, while satisfying orthogonality and zero moment conditions, as will be shown.

For the two-channel case, the shortest filters for which the filter bank is orthogonal and having K zero moments, are the well known filters described by Daubechies [5]. For $K = 2$ zero moments, those filters $H(z)$ and $F(z)$ are of length 4. For this system, designated D_2, the iterated filters in Figure 1 are of length 10 and 4. Without the constraint that the filters are products, an orthogonal filter bank with $K = 2$ zero moments can be obtained where the filter lengths are 8 and 4, as shown in Figure 2, side by side with the iterated D_2 system. That reduction of two samples grows with the number of stages, as in Figure 3. We make several comments regarding Figures 2 and 3.

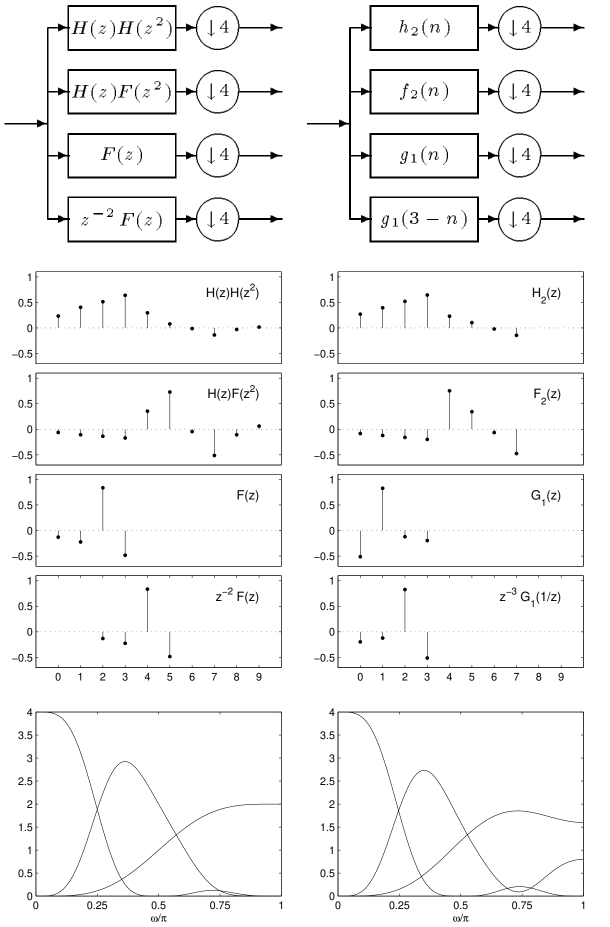

Figure 2: Comparison of iterated D_2 filter bank (left-hand side) and slantlent filter bank (right-hand side).

1. Each filter bank (equivalently, discrete-time basis) is orthogonal. The filters in the synthesis filter bank are obtained by time-reversal of the analysis filters.
2. Each filter bank has 2 zero moments. The filters (except for the lowpass ones) annihilate discrete-time polynomials of degree less than 2.
3. Each filter bank has an octave-band characteristic.
4. The scale-dilation factor is 2 for each filter bank. Between scales, the filters dilate by roughly a factor of 2. (In the slantlet filter banks, by exactly a factor of 2.)
5. Each filter bank provides a multiresolution decomposition. By discarding the highpass channels, and passing only the lowpass channel outputs through the synthesis filter bank, a lower resolution version of the original signal is obtained.
6. The slantlet filter bank is less frequency selective than the traditional DWT filter bank, due to the shorter length of the filters. The time-localization is improved with a degradation of frequency selectivity.
7. The slantlet filters are piecewise linear.

It must be admitted that, although both types of filter banks posses the same number of zero moments, the smoothness properties of the filters are somewhat different. In Figures 2 and 3, the

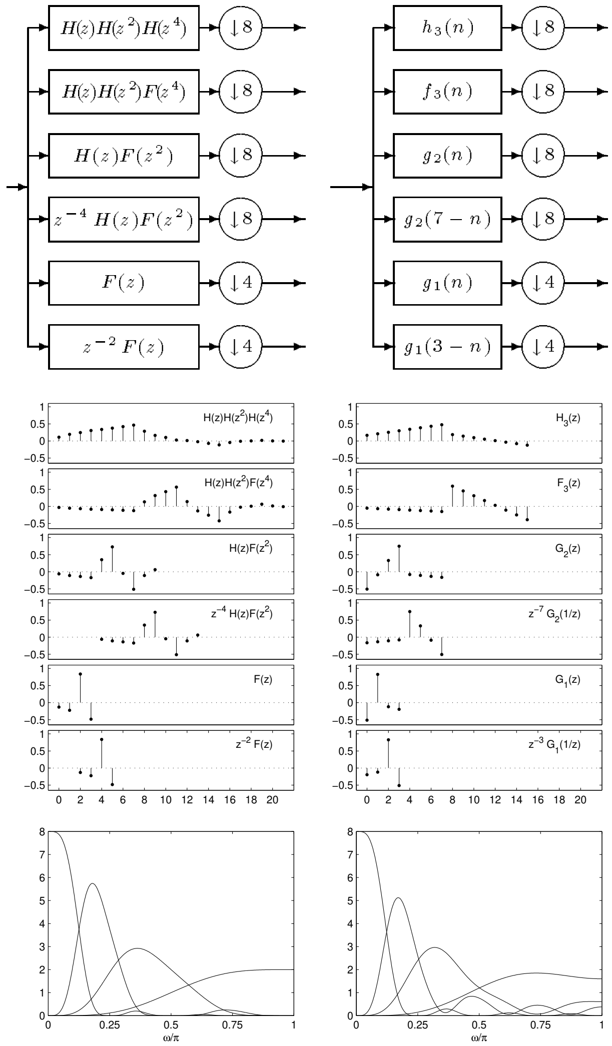

Figure 3: Comparison of iterated D_2 filter bank (left-hand side) and slantlet filter bank (right-hand side).

slantlet filters have greater "jumps" than do the iterated D_2 filters. However, the Haar basis, with its discontinuities, is suitable for analyzing piecewise constant functions that have jumps. Likewise, the slantlet filter bank appears appropriate for the analysis of piecewise linear functions, as illustrated in the denoising example below. The ability to model jumps is also relevant for other applications, like edge detection and change point analysis, in which the detection of abrupt changes in an otherwise relatively smooth but unknown function is considered [9].

2.1. Notation

We will call the scale with which $g_i(n)$, $f_i(n)$, and $h_i(n)$ analyze a signal, scale i. The length of the filters for scale i will be proportional to 2^i. That is approximately true for iterated filter banks, however, it is exact for the filter banks introduced in this paper. In general, the support of $g_i(n)$, $f_i(n)$ and $h_i(n)$ will be 2^{i+1}.

We should clarify the way in which the slantlet filter banks in Figures 2 and 3 are generalized to l-scales. That is done as follows.

The l-scale filter bank has $2l$ channels. The lowpass filter is to be called $h_l(n)$. The filter adjacent to the lowpass channel is to be called $f_l(n)$. Both $h_l(n)$ and $f_l(n)$ are to be followed by downsampling by 2^l. The remaining $2l-2$ channels are filtered by $g_i(n)$ and its shifted time-reverse for $i = 1, \ldots, l-1$. Each is to be followed by downsampling by 2^{i+1}. It follows that the filter bank is critically sampled.

2.2. Derivations

That the sought filters $g_i(n)$, $f_i(n)$, and $h_i(n)$ are piecewise linear, is central to the following derivation. When coupled with the 2 zero moments, it simplifies orthogonality conditions: First, suppose that $g_i(n)$, $f_i(n)$, and $h_i(n)$ are each linear over the interval $n \in \{0, \ldots, 2^i-1\}$ and over the interval $n \in \{2^i, \ldots, 2^{i+1}-1\}$, as in Figures 2 and 3. Suppose also that $g_i(n)$ and $f_i(n)$ have two zero moments, as filters, they annihilate "ramps". With $j > i$, over the support of $g_i(n)$ the functions $g_j(n)$, $f_j(n)$, and $h_j(n)$ are linear, so orthogonality between scales is immediate.

Because the sought filter $g_i(n)$ is to be linear over the two above-mentioned intervals, it is described by four parameters, and can be written as

$$g_i(n) = \begin{cases} a_0 + a_1\, n & \text{for } n = 0, \ldots 2^i - 1 \\ b_0 + b_1\,(n - 2^i) & \text{for } n = 2^i, \ldots 2^{i+1} - 1 \end{cases}$$

Therefore, to obtain $g_i(n)$ such that the sought l-scale filter bank is orthogonal with 2 zero moments, requires obtaining parameters a_0, a_1, b_0, b_1 so that

1. $g_i(n)$ is of unit norm. $\displaystyle\sum_{n=0}^{2^{i+1}-1} g_i^2(n) = 1$
2. $g_i(n)$ is orthogonal to its shifted time-reverse.
$$\sum_{n=0}^{2^{i+1}-1} g_i(n)\, g_i(2^{i+1} - 1 - n) = 0$$
3. $g_i(n)$ annihilates linear discrete time polynomials.
$$\sum_{n=0}^{2^{i+1}-1} g_i(n) = 0, \qquad \sum_{n=0}^{2^{i+1}-1} n\, g_i(n) = 0$$

Each of the conditions can be written as an algebraic equation in terms of the four parameters a_0, a_1, b_0, b_1, to obtain a multivariate polynomial system of equations. The conditions are nonlinear in the four parameters, however, with assistance from the computer algebra systems *Maple* and *Singular* [8] we obtain the following expressions for $g_i(n)$.

$$g_i(n) = \begin{cases} a_0 + a_1\, n & \text{for } n = 0, \ldots 2^i - 1 \\ b_0 + b_1\,(n - 2^i) & \text{for } n = 2^i, \ldots 2^{i+1} - 1 \end{cases}$$

where

$$\begin{aligned}
m &= 2^i \\
s_1 &= 6\sqrt{m/((m^2-1)(4\,m^2-1))} \\
t_1 &= 2\sqrt{3/(m\cdot(m^2-1))} \\
s_0 &= -s_1\cdot(m-1)/2 \\
t_0 &= ((m+1)\cdot s_1/3 - m\,t_1)(m-1)/(2\,m) \\
a_0 &= (s_0 + t_0)/2 \\
b_0 &= (s_0 - t_0)/2 \\
a_1 &= (s_1 + t_1)/2 \\
b_1 &= (s_1 - t_1)/2
\end{aligned}$$

The same approach works for $f_i(n)$ and $h_i(n)$. By using again a piecewise linear form, and expressing the orthogonality and moment conditions as a multivariate polynomial system, we obtain the following.

$$h_i(n) = \begin{cases} a_0 + a_1\, n & \text{for } n = 0, \ldots 2^i - 1 \\ b_0 + b_1\,(n - 2^i) & \text{for } n = 2^i, \ldots 2^{i+1} - 1 \end{cases}$$

$$f_i(n) = \begin{cases} c_0 + c_1\, n & \text{for } n = 0, \ldots 2^i - 1 \\ d_0 + d_1\,(n - 2^i) & \text{for } n = 2^i, \ldots 2^{i+1} - 1 \end{cases}$$

where

$$\begin{aligned}
m &= 2^i \\
u &= 1/\sqrt{m} \\
v &= \sqrt{(2\,m^2+1)/3} \\
a_0 &= u\cdot(v+1)/(2\,m) \\
b_0 &= u\cdot(2\,m - v - 1)/(2\,m) \\
a_1 &= u/m \\
b_1 &= -a_1 \\
q &= \sqrt{3/(m\cdot(m^2-1))}/m \\
c_1 &= q\cdot(v-m) \\
d_1 &= -q\cdot(v+m) \\
d_0 &= d_1\cdot(v+1-2\,m)/2 \\
c_0 &= c_1\cdot(v+1)/2
\end{aligned}$$

We note that $h_i(n)$ and $f_i(n)$ specialize to the Daubechies length-4 filters for $i = 1$, as expected.

We also wish to note that a relationship exists between the bases described in this paper, and the general K-regular M-band scaling filters described in [11, 12]. Namely, for $K = 2$, the maximally regular lowpass filters of [11], and the lowpass filter $h_i(n)$ above, are identical. However, the M-band systems described in [11] provide a different tiling of the time-frequency plane that might better suite certain applications, and its intention is not the improvement of time-localization properties while preserving the essential time-frequency tiling of the 2-band DWT.

2.3. Support Length

In Figures 2 and 3 it was seen that the support of the slantlet filters is less than those of the filters obtained by filter bank iteration. It is interesting to note the difference for the general l-scale case. The iterated filter bank, with Daubechies length-4 filters, analyzes scale i with a filter of length $3\cdot 2^i - 2$. On the other hand, the slantlet filter bank analyzes scale i with the filter $g_i(n)$ of length 2^{i+1}. That gives a reduction of $2^i - 2$ samples for scale i. The ratio tends to 2/3 as i increases (for coarser scales). That reduction in the support of the analysis filters is precisely what was sought.

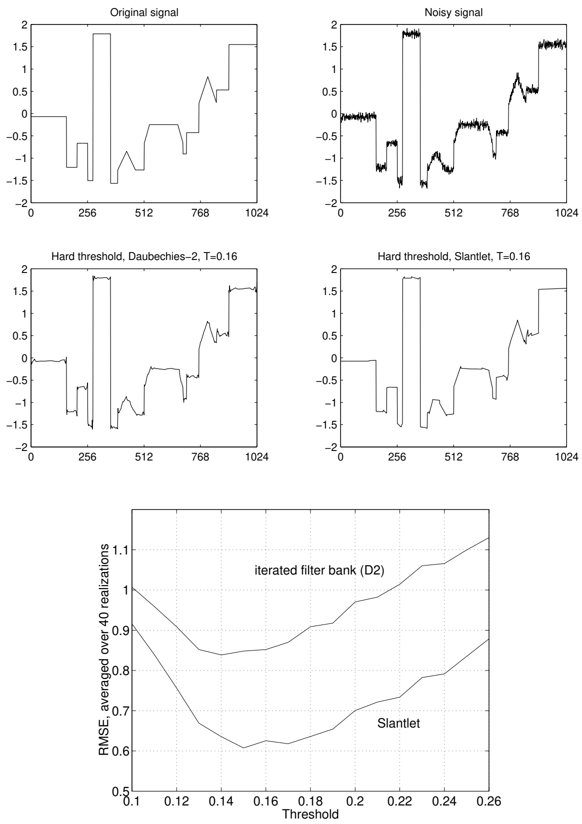

Figure 4: Denoising via hard thresholding with the iterated D_2 filter bank and the slantlet DWT.

2.4. Comparison to Slant Transform

Interestingly, the Haar basis can be obtained by downsampling the Walsh basis. Both are piecewise constant, but the Walsh transform serves as a minimal complexity DCT for frequency analysis, while the Haar transform gives a multiresolution decomposition. A piecewise linear basis, that follows the spirit of the Walsh transform, is the *slant* transform [1, 7, 10]. In a loose sense, the transform described in this paper is to the slant transform, what the Haar transform is to the Walsh transform. The analogy is only loose, however, their similarities suggest the name *slantlet* transform for the transform described in this paper.

2.5. Denoising Example

In this example, the slantlet and iterated D_2 transforms are compared, using hard thresholding in the transform domain. Figure 4 illustrates the results. Denoising with the slantlet transform yields the same artifacts and noise spikes, but for this example, generally they were reduced. Varying the threshold used, and averaging over 40 realizations for each threshold, the curve illustrating the root-mean-square error in Figure 4 was obtained. That figure shows that for this example the slantlet transform gives an improvement over the iterated D_2 filter bank. It indicates, that on average, for thresholds between 0.12 and 0.24, the error with the slantlet is smaller than that obtained with D_2 for any threshold. That a wider choice of thresholds gives such results is important, for in practice of course, the best threshold for a particular example is unknown.

3. CONCLUSION

The smoothing of data while preserving edges relatively well is an essential advantage of wavelets in denoising and it depends in part on both the short support of the basis functions with respect to their scale, and the number of zero moments. This paper describes an orthogonal filter bank for the discrete wavelet transform with two zero moments, where the filters are of shorter support than those of the iterated D_2 filter bank tree.

Although not based on an iterated filter bank tree, the filter bank introduced in this paper retains the main desirable characteristics of the usual DWT filter bank, namely: orthogonality, an octave-band characteristic, a scale-dilation factor of 2, and an efficient implementation. Due to space limitations, we can only state here that an efficient implementation exists.

Matlab programs for the slantlet transform, its inverse, and a shift-invariant (redundant) variant, are available from the author.

	Iterated D_2	Slantlet
Orthogonal	y	y
Octave-band	y	y
Zero moments	2	2
Efficient implementation	y	y
Piecewise linear	n	y
Support at scale i	$3 \cdot 2^i - 2$	2^{i+1}

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