Balanced Multiwavelet Bases Based on Symmetric FIR Filters

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***Abstract*—This paper describes a basic difference between multiwavelets and scalar wavelets that explains, without using zero moment properties, why certain complications arise in the implementation of discrete multiwavelet transforms. Assuming we wish to avoid the use of prefilters in implementing the discrete multiwavelet transform, it is suggested that the behavior of the iterated filter bank associated with a multiwavelet basis of multiplicity r is more fully revealed by an expanded set of r^2 scaling functions $\phi_{i,j}$. This paper also introduces new K-balanced orthogonal multiwavelet bases based on symmetric FIR filters. The nonlinear design equations arising in this work are solved using the Gröbner basis. The minimal-length K-balanced multiwavelet bases based on even-length symmetric FIR filters are better behaved than those based on odd-length symmetric FIR filters, as illustrated by special relations they satisfy and by examples constructed.**

***Index Terms*—Filter banks, multiwavelet bases, wavelet transforms.**

I. Introduction

An IMPORTANT motivation for the study of *multi*wavelet bases is the design of orthogonal FIR filter banks with symmetry properties — a famous limitation of scalar wavelets.[1] The remarkable symmetric GHM scaling functions of [7] triggered a great interest in multiwavelets since symmetry is important in image processing: one of the primary areas of wavelet application. However, there exist important differences between multiwavelet and scalar wavelet bases, and these differences become apparent when the discrete multiwavelet transform is implemented. Indeed, in the processing of discrete-time signals, complications arise for multiwavelet transforms that do not arise for scalar wavelet transforms. The usual correction for this is the specialized preprocessing (prefiltering) of the discrete-time data, which sometimes destroys the very properties a multiwavelet basis is designed to have.

Previous papers [12], [16] have highlighted an important difference between multiwavelet and scalar wavelet bases. For multiwavelet bases, they focused on zero moment properties of the associated filter bank and the construction of specialized K*-balanced* multiwavelets. In the following, we emphasize an even more basic difference between multiwavelets and scalar wavelets that explains, without using zero moment properties, why certain complications arise in the implementation of multiwavelet transforms.

In practice, wavelet transforms are usually implemented as iterated filter-bank trees. The scaling and wavelet functions ϕ and ψ are useful because they reflect the behavior of the iterated filter bank. For scalar wavelet transforms, after only a few iterations of the filter bank, the iterated filters closely resemble the scaling and wavelet functions. However, the iterated filterbank tree associated with a multiwavelet transform behaves quite differently from one associated with a scalar wavelet transform.

For a multiwavelet basis based on r scaling functions and r wavelet functions, we claim that these $2r$ functions by themselves do not accurately reveal the full behavior of the associated filter bank. We suggest that by looking only at the $2r$ functions, it is hard to evaluate how effective the filter bank is for signal processing applications. We propose that the behavior of the iterated filter bank associated with the basis is better revealed by a set of r^2 scaling functions and r^2 wavelet functions.

This paper also introduces new K-balanced orthogonal multiwavelet bases based on symmetric FIR filters. Both odd-length and even-length FIR filters are considered. The nonlinear design equations arising in this work are solved using the Gröbner basis. The complications that arise for multiwavelet bases, in general, are suppressed for balanced multiwavelets based on even-length symmetric FIR filters.

II. Preliminaries

This paper considers multiwavelet bases based on two scaling functions $\phi_0(t)$, $\phi_1(t)$ and two wavelet functions $\psi_0(t)$, $\psi_1(t)$. Accordingly, there are two scaling filters $h_0(n)$, $h_1(n)$ and two wavelet filters $h_2(n), h_3(n)$. Throughout, $i, j, k, m, n \in$ Z and $t \in$ R.

The functions $\phi_0(t)$, $\phi_1(t)$ are *orthogonal multiscaling functions* if we have the following.

1) $\phi_0(t)$ and $\phi_1(t)$ satisfy a *matrix* dilation equation

$$\underline{\phi}(t) = \sqrt{2} \sum_n C(n) \underline{\phi}(2t-n) \quad (1)$$

where $\underline{\phi}(t) = (\phi_0(t), \phi_1(t))^t$, and $C(n)$ are 2 × 2 matrices.

2) $\phi_0(t)$ and $\phi_1(t)$are orthogonal to their integer shifts.

$$\int \phi_i(t)\phi_j(t-n)\, dt = \delta(i-j) \cdot \delta(n).$$

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[1]To distinguish multiwavelet bases from wavelet bases based on a single scaling function, we will call the latter *scalar* wavelet bases.

The notation for $C(n)$ used in this paper is $[C(n)]_{i,j} = h_i(2n+j)$. For example

$$C(0) = \begin{pmatrix} h_0(0) & h_0(1) \\ h_1(0) & h_1(1) \end{pmatrix}, \quad C(1) = \begin{pmatrix} h_0(2) & h_0(3) \\ h_1(2) & h_1(3) \end{pmatrix}$$

etc. We also write $C(n) = (\underline{h}(2n) \;\; \underline{h}(2n+1))$, where $\underline{h}(n) = (h_0(n), h_1(n))^t$.

Samples of $\phi_i(t)$ can be calculated as in the scalar wavelet case. The value of $\phi_i(t)$ on the integers can be computed by solving an eigenvalue problem, and then, the value of $\phi_i(t)$ on the dyadics ($t = k/2^j$) can be computed by recursively using the dilation equation [1], [19]. This is the method used to plot all scaling and wavelet functions in this paper.

For h_0, h_1 to generate orthogonal scaling functions ϕ_0, ϕ_1, it is necessary that

$$\sum_n h_i(n) h_j(n+4k) = \delta(i-j) \cdot \delta(k). \tag{2}$$

This condition characterizes orthogonal four-channel filter banks and arises here because a two-channel vector-filter bank can be redrawn as a four-channel scalar filter bank with interleaving of subband signals [15]. (Accordingly, lattice parameterizations for orthonormal multiwavelets can be obtained from those for multichannel filter banks [10].) This form of the filter bank, which is illustrated in Fig. 1, is a useful reorganization of the signal flowgraph. Because of the interleaving, the filters h_0 and h_1 must be jointly designed and/or preprocessing of the input $x(n)$ is needed.

III. Shift Properties

Given a filter bank, one reason it is useful to view the associated wavelet and scaling functions is that they allow evaluation and interpretation of the behavior of the filterbank under iteration. By inspecting the characteristics of the scaling and wavelet functions ϕ and ψ, we may draw conclusions about how the discrete-time filters h_i behave when they are used in an iterated filterbank tree. However, this is only possible in so far as the scaling and wavelet functions accurately reflect the behavior of the iterated filters. For *scalar* wavelet bases, ϕ and ψ give a very good indication of the iterated filters. The iterated filters resemble the scaling and wavelet functions after only a few iterations. The definition of the functions ϕ and ψ is an excellent tool for the design of filters that are to be employed in iterated filter banks.

However, for *multi*wavelet bases, the two scaling functions and the two wavelet functions do not fully reveal the behavior of the iterated filterbank tree. To clarify this difference between scalar wavelets and multiwavelets, we begin with a simple question about scalar wavelet bases. If

$$\phi(t) = \sqrt{2} \sum_n h(n) \phi(2t-n)$$

then how does $\phi(t)$ change when one shifts $h(n)$? Specifically, let

$$u(t) = \sqrt{2} \sum_n h(n-L) u(2t-n).$$

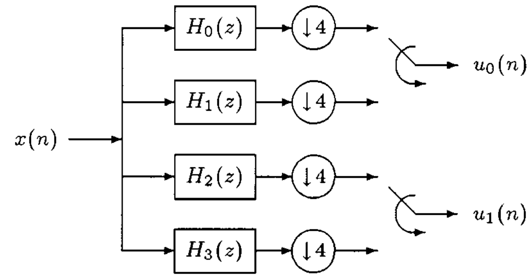

Fig. 1. Multiwavelet filter bank ($r = 2$) drawn as a scalar four-channel filter bank with interleaving.

How is $u(t)$ related to $\phi(t)$? It is not difficult to prove that $u(t) = \phi(t-L)$. That is, shifting $h(n)$ by L also shifts $\phi(t)$ by L. It makes sense that shifting the filter h has no real effect on the nature of ϕ. After all, in a filter bank implementation of the scalar-DWT, shifting the filter $h(n)$ is equivalent to shifting the input sequence $x(n)$. The operation of the filter $h(n-1)$ with input $x(n)$ is equivalent to the operation of the filter $h(n)$ with input $x(n-1)$. However, for multiwavelet filter banks, the situation is quite different.

To distinguish between the scaling functions generated by the differing shifts of the filters h_i, we introduce the following notation:

$$\begin{pmatrix} \phi_{0,0}(t) \\ \phi_{1,0}(t) \end{pmatrix} = \sqrt{2} \sum_n \begin{pmatrix} h_0(2n) & h_0(2n+1) \\ h_1(2n) & h_1(2n+1) \end{pmatrix} \cdot \begin{pmatrix} \phi_{0,0}(2t-n) \\ \phi_{1,0}(2t-n) \end{pmatrix}$$

$$\begin{pmatrix} \phi_{0,1}(t) \\ \phi_{1,1}(t) \end{pmatrix} = \sqrt{2} \sum_n \begin{pmatrix} h_0(2n-1) & h_0(2n) \\ h_1(2n-1) & h_1(2n) \end{pmatrix} \cdot \begin{pmatrix} \phi_{0,1}(2t-n) \\ \phi_{1,1}(2t-n) \end{pmatrix}.$$

That is, $\phi_{i,0}(t)$ represents the scaling functions generated by the unshifted filters $\underline{h}(n)$, whereas $\phi_{i,1}(t)$ represents the scaling functions generated by the shifted filters $\underline{h}(n-1)$. Note that shifting all the filters $h_i(n)$ in the filter bank of Fig. 1 by the same amount affects neither the orthogonality of the filter bank nor the orthogonality of the multiwavelet basis it generates.

In the multiwavelet case, the scaling functions generated by $\underline{h}(n)$ and those generated by $\underline{h}(n-1)$ can be totally different. This is best illustrated by example. In Fig. 2, the classic GHM scaling functions are shown, along with the scaling functions generated when the GHM scaling filters are shifted by one sample. Although $\phi_{i,0}$ are continuous and differential except at a few points, $\phi_{i,1}$ are extremely nonsmooth. The same behavior is evident for other unbalanced multiwavelet bases. Note that if the filters are shifted by an *even* number of samples, then the scaling functions *are* simply shifted; for this reason, we consider only the scaling functions generated by $\underline{h}(n)$ and $\underline{h}(n-1)$.

From this example, we conclude that $\phi_{i,0}$ alone does not reflect the full nature of the iterated multiwavelet filterbank tree. We should look at $\phi_{i,1}$ as well. Even if the two scaling functions $\phi_{0,0}$ and $\phi_{1,0}$ are highly differentiable, it is possible that the functions $\phi_{0,1}$ and $\phi_{1,1}$ are highly discontinuous, in which case, the filterbank must be preceded by a properly designed

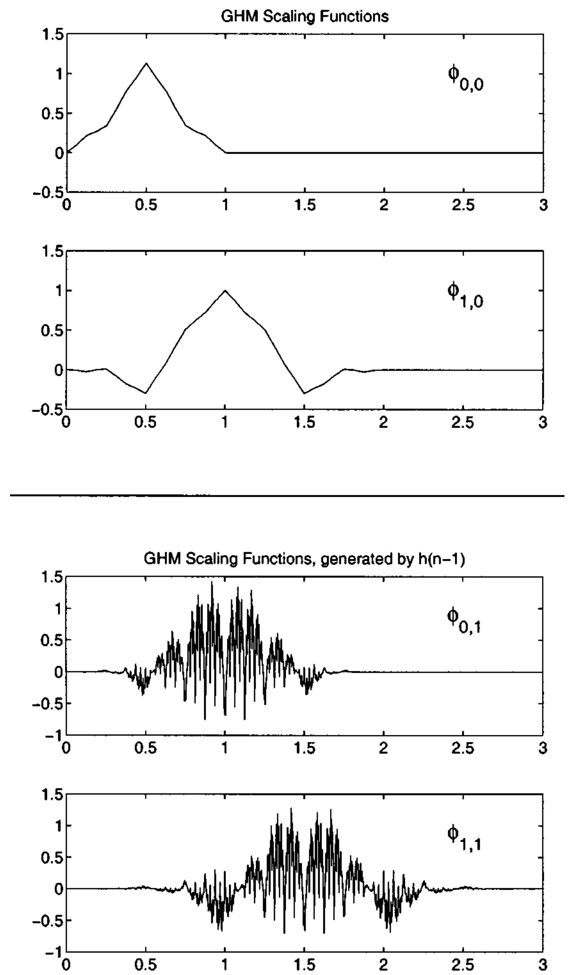

Fig. 2. GHM scaling functions generated by $\underline{h}(n)$ and $\underline{h}(n-1)$.

prefilter. Otherwise, the iterated filterbank structure will not be effective for signal processing applications. The nonsmooth nature of $\phi_{i,1}$ for certain multiwavelet systems is one way of explaining why prefilters are important in the implementation of discrete multiwavelet transforms.

In previous papers, [12], [16], the design of specialized (K-balanced) multiwavelets that do not require prefiltering was considered. However, in those papers, the functions $\phi_{i,1}(t)$ were not considered. In the following, our goal is again the design of multiwavelet filter banks that do not require a specialized preprocessing step. In this paper, we suggest that to evaluate the effectiveness of the iterated filterbank tree, it is informative to inspect not only $\phi_{i,0}(t)$ but the functions $\phi_{i,1}(t)$ as well. The functions $\phi_{i,1}$ are relevant because the basic behavior of the filterbank should not depend on the shift of the filters $h_i(n)$, just as it should not depend on the shift of the input $x(n)$.

A. Remarks

Shifting the filters $h_i(n)$ does not always destroy/degrade the regularity of $\phi_i(t)$. For example, a scalar wavelet basis can always be embedded in the multiwavelet framework. If $h_1(n) = h_0(n-2)$, then all the scaling functions are simply shifted versions of a single scaling function $\phi(t)$

$$\phi_{0,0}(t) = \phi(2t) \qquad \phi_{0,1}(t) = \phi(2t-1)$$
$$\phi_{1,0}(t) = \phi(2t-1) \qquad \phi_{1,1}(t) = \phi(2t-2).$$

Recalling the importance of symmetry, the question becomes the following: Do there exist orthogonal multiwavelet bases based on symmetric FIR filters for which all four scaling functions $\phi_{i,j}(t)$ have acceptable smoothness properties? The K-balanced multiwavelet bases by Lebrun and Vetterli [12] are examples for which each of the four scaling functions are highly regular. They are based not entirely on symmetric scaling filters; instead, $h_1(n) = h_0(N-n)$, $h_2(n) = h_2(N-n)$, and $h_3(n) = -h_3(N-n)$. The scaling filters are flipped versions of one another, whereas the wavelet filters are symmetric/anti-symmetric, respectively, which are properties that carry over to the scaling and wavelet functions $\phi_{i,0}(t)$ and $\psi_{i,0}(t)$ [and to $\phi_{i,1}(t)$ and $\psi_{i,1}(t)$].

IV. Balance Order

For scalar wavelet bases, the associated filter bank inherits the zero moment properties of $\psi(t)$. However, for multiwavelet bases in general, the associated filter bank does not [12], [16]. Multiwavelet bases for which the zero moment properties *do* carry over to the discrete-time filter bank are called *balanced* after Lebrun and Vetterli [11], [12]. Specifically, multiwavelet bases for which the associated filter bank preserves/annihilates the set $\mathcal{P}_{K-1}$ of polynomials of degree $k < K$ are said to be *order-K balanced*; see [11], [12], and [16]. The condition for K-balancing is

$$(z^{-3} + z^{-2} + z^{-1} + 1)^K \,|\, [H_0(z) + Q_K(z^4) H_1(z)] \qquad (3)$$

where

- $Q_1(z) = 1$;
- $Q_2(z) = (3 - z^{-1})/2$;
- $Q_3(z) = (15 - 10\, z^{-1} + 3\, z^{-2})/8$.

See [12] and [16] for general K. The examples to be given in Section V will be balanced up to their approximation order. Some of these issues are also addressed, in a different approach, in [22].

V. K-Balanced Multiwavelets Based on Symmetric FIR Filters

The GHM scaling functions $\phi_{i,0}$ illustrated in Fig. 2 are based on symmetric scaling filters h_0 and h_1 of lengths 3 and 7, respectively. A pair of wavelets orthogonal to $\phi_{i,0}$ was given in [4] and [20]. However, although the GHM multiwavelet basis has two zero moments, it is not balanced. The integral of $\phi_{0,0}$ and the integral of $\phi_{1,0}$ are not equal, and neither are the sums of h_0 and h_1. The GHM system is zero balanced. In this section, we investigate the design of "K-balanced" versions of the GHM basis.

We also look at all four functions $\phi_{i,j}$ and not just $\phi_{i,0}$. Specifically, we seek to design symmetric FIR filters h_0 and h_1 that generate orthogonal K-balanced multiwavelet bases. That is, h_0 and h_1 must satisfy the nonlinear orthogonal constraints given by (2) and the balancing conditions given by (3). We seek the shortest filters satisfying these equations so that the equations totally define the filters up to a finite number of solutions—there are no continuously-variable free parameters remaining.

As mentioned above, we use Gröbner bases to obtain solutions to the nonlinear system of equations—sometimes obtaining explicit solutions in terms of radicals. In a sense, Gröbner bases extend Gaussian elimination to multivariate polynomial systems [2]. Gröbner bases have also been used for filter design in [6], [12], [14], and [18], for example. We used the software *Singular* [8] to carry out the Gröbner basis computations.

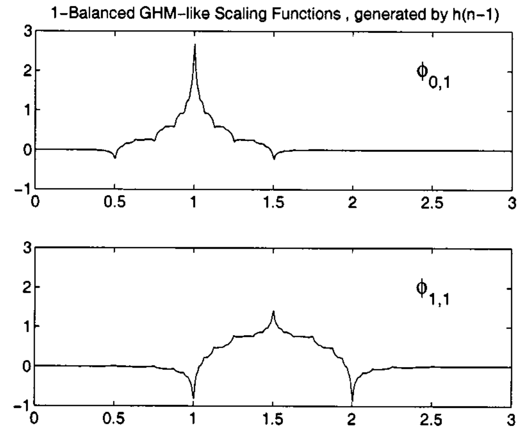

Fig. 3. One-Balanced GHM-like scaling functions $\phi_{i,1}(t)$ generated by $\underline{h}(n-1)$. The functions $\phi_{i,0}(t)$ generated by $\underline{h}(n)$ are illustrated in [17].

A. Odd-Length Case

This section describes GHM-like multiwavelet bases balanced up to order K. Like the GHM basis, we have the following.

1) All $\phi_{i,j}(t)$ are symmetric.
2) Both $h_0(n)$ and $h_1(n)$ are odd-length symmetric (Type 1) FIR filters.
3) $h_0(n)$ and $h_1(n)$ differ in length by 4.

For none of the one–, two-, and three-balanced minimal-length orthogonal multiwavelets bases, based on odd-length symmetric FIR filters, are both $\phi_{i,0}$ and $\phi_{i,1}$ acceptably smooth. That suggests that perhaps symmetric *odd*-length filters are somehow not the most compatible with "smooth" K-balanced orthogonal multiwavelet systems. For balanced multiwavelets based on *even*-length symmetric filters, the results are more positive.

One-Balanced Solutions: It was found in [17] that the minimal-length 1-balanced solution, based on odd-length symmetric FIR filters, are of lengths 3 and 7, with h_0, h_1 supported on $n = (0, 1, 2)$ and $n = (0, \cdots, 6)$, respectively. While the scaling functions $\phi_{i,0}(t)$ (which are illustrated in [17]) are free of cusps, the scaling functions $\phi_{i,1}(t)$, which are illustrated in Fig. 3, do have cusps, making them very poor for discrete-time signal processing applications. This helps explain the poor compression results reported in [24] for this particular multiwavelet system. When processing is performed on the wavelet coefficients (quantization or thresholding, for example), the nonsmooth behavior will become apparent in the processed signal.

Two-Balanced Solutions: By increasing the lengths to (7, 11), it is possible to obtain a two-balanced GHM-like multiwavelet basis. The filters h_0 and h_1 are supported on $n = (0, \cdots, 6)$ and $n = (0, \cdots, 11)$, respectively. Unfortunately, the scaling functions $\phi_{i,0}(t)$,which are illustrated in [17], have sharp cusps, making them very poor for signal processing applications. Although they are not shown, the scaling functions $\phi_{i,1}(t)$ closely resemble $\phi_{i,0}(t)$ for this example and likewise have sharp cusps.

Three-Balanced Solutions: Using Gröbner bases, we also constructed all the minimal-length three-balanced GHM-like

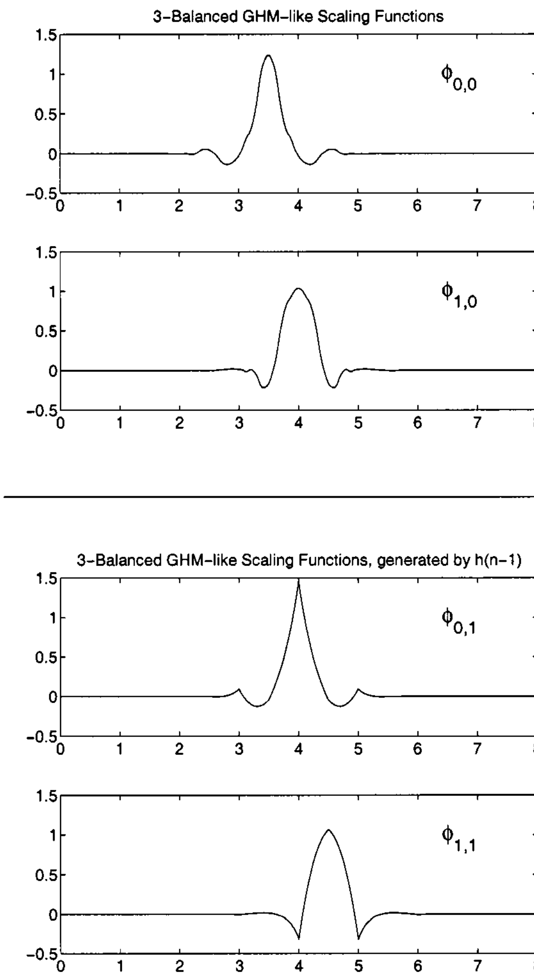

Fig. 4. Three-balanced GHM-like scaling functions generated by $\underline{h}(n)$ and $\underline{h}(n-1)$.

multiwavelet bases. Noting that the one-balanced basis was based on h_0 and h_1 of lengths 3 and 7 and that the two-balanced basis was based on h_0 and h_1 of lengths 7 and 11, we expected to obtain three-balanced bases for lengths 11 and 15. However, for these lengths, the use of Gröbner bases revealed that the shortest possible lengths for order-3 balancing was 15 and 19, with filters h_0 and h_1 supported on $n = (0, \cdots, 14)$ and

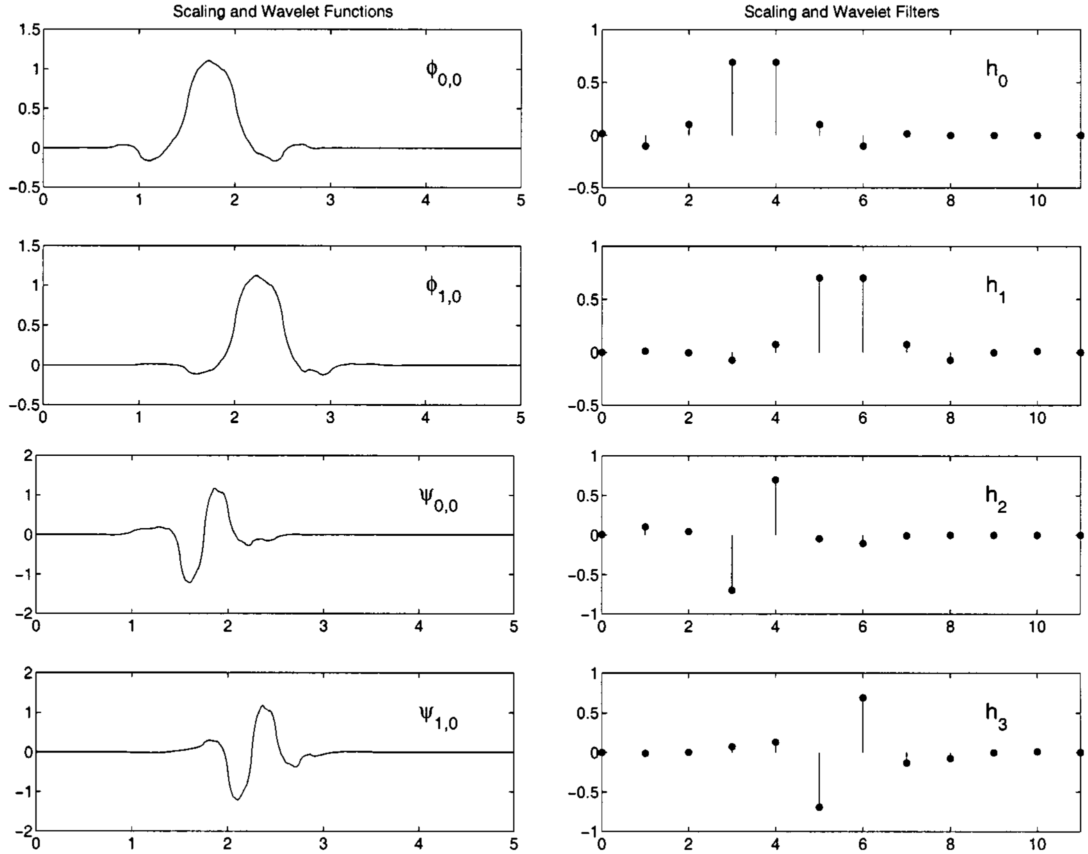

Fig. 5. Two-balanced orthogonal multiwavelet basis based on even-length symmetric FIR filters.

$n = (0, \cdots, 18)$, respectively. The calculation of the lexical Gröbner basis with exact integer arithmetic, which took four weeks on a 200-MHz Sun Ultra 2, revealed that (counting multiplicities) there are 128 roots to the multivariate polynomial system of design equations. However, further computations revealed that there are only 16 distinct solutions—each root being repeated eight times. Of the 16 solutions, Fig. 4 illustrates the smoothest scaling functions $\phi_{i,0}$. Unfortunately, even though $\phi_{i,0}(t)$ is reasonably smooth, $\phi_{i,1}(t)$ has sharp cusps, making this system poor for signal processing applications. Fig. 4 shows that even if the system is balanced, it does not necessarily mean that $\phi_{i,0}$ and $\phi_{i,1}$ have the same degree of smoothness.

B. Even-Length Case

This section describes orthogonal multiwavelet scaling functions for which we have the following.

1) Both $h_0(n)$ and $h_1(n)$ are even-length symmetric (Type 2) FIR filters.
2) $h_0(n)$ and $h_1(n)$ differ in length by 4.

Note that the Haar basis is a special case of a multiwavelet bases based on even-length symmetric filters. Letting $h_0(n) = h_{HAAR}(n)$ and $h_1(n) = h_{\mathrm{HAAR}}(n-2)$ yields the Haar basis as a one-balanced multiwavelet basis. (Multiwavelets bases based on odd-length symmetric FIR filters do not specialize to the Haar basis.) Therefore, we examine two- and three-balanced solutions.

Symmetries: For even-length symmetric scaling filters h_i, the scaling functions $\phi_{i,0}(t)$ are not exactly symmetric. Although this is initially surprising, it is explained simply by observing the polyphase components $h_i(2n)$ and $h_i(2n+1)$. For an odd-length symmetric filter $h(n)$, both $h(2n)$ and $h(2n+1)$ are symmetric — one polyphase component is even-length symmetric, and the other is odd-length symmetric. However, for an even-length symmetric filter $h(n)$, neither $h(2n)$ nor $h(2n+1)$ is symmetric in general. This difference between even- and odd-length symmetry leads to the lack of perfect symmetry for the even-length case. However, for even-length symmetric $h_0(n)$, $h_1(n)$, the scaling functions $\phi_{i,0}$ and $\phi_{i,1}$ are related through a simple time-reversal and shift.

$$\phi_{0,1}(t) = \phi_{0,0}(L-t) \tag{4}$$

and

$$\phi_{1,1}(t) = \phi_{1,0}(L+1-t) \tag{5}$$

where h_0 and h_1 are of lengths $2L$ and $2L+4$, respectively [supported on $n = (0, \cdots, 2L-1)$ and $n = (0, \cdots, 2L+3)$, respectively]. The relations (4) and (5) are proven in Appendix A, but they can be explained informally by again observing the polyphase components. For an even-length symmetric filter $h(n)$, the components $h(2n)$ and $h(2n+1)$ are related through a flip; $h(2n+1)$ is simply a time-reversed shifted version of $h(2n)$. Therefore, when the scaling filters $h_i(n)$ are shifted by a single sample, they generate the same scaling functions, except for a time-reversal and shift. That is a desirable property—both $\phi_{i,0}$ and $\phi_{i,1}$ have exactly the same degree of smoothness. Accordingly, the smoothness of the discrete-time basis functions associated with the discrete multiwavelet transform (implemented without prefilters) does not depend on the shift. Provided $\phi_{i,0}$ is smooth, when implementing the discrete multiwavelet transform, preprocessing discrete-time data is not needed to compensate for the variance of the smoothness with respect to the shift.

For the even-length case, it also turns out that the polyphase components of h_i must be CQF filters, as shown in Appendix B.

Two-Balanced Solutions: For order-2 balancing, the minimal lengths of h_0 and h_1 are 8 and 12, supported on $n = (0, \cdots, 7)$ and $n = (0, \cdots, 11)$, respectively. Via Gröbner bases, we found that there are eight distinct solutions to the defining nonlinear equations. The best solution (in terms of smoothness) is shown in Fig. 5; the coefficients are given in Table I, as in (5a), shown at the bottom of the page. We also show in the figure a set of wavelets that are orthogonal to $\phi_{i,0}$, which are based on antisymmetric even-length FIR filters h_2 and h_3 supported on $n = (0, \cdots, 7)$ and $n = (0, \cdots, 11)$, respectively. The scaling functions $\phi_{i,1}$ are not shown because they are exactly the same as $\phi_{i,0}$ up to a shift and time-reverse as explained above. Note that while the scaling functions are not exactly symmetric, they are nearly so. In addition, note that they do not have cusps as do the solutions in the odd-length case. The zero at $\omega = \pi$, which an even-length symmetric FIR filter must have, may contribute the greater degree of smoothness. The odd-length filters above do not have zeros at $\omega = \pi$. This basis resembles the Haar basis in that both the scaling filters are symmetric, and both the wavelet filters are antisymmetric, in contrast with several other multiwavelet bases.

Three-Balanced Solutions: We obtained minimal-length three-balanced solutions h_0 and h_1 of lengths 12 and 16, respectively. The smoothest three-balanced solution closely resembles the two-balanced solution shown in Fig. 5.

VI. Conclusion

Multiwavelets became a focus of research partly because they made possible the construction of wavelet systems that are simultaneously orthogonal, symmetric, and FIR. However, it became clear that the implementation of the discrete multiwavelet transform required the design of specialized prefilters; see, for example [5], [9], [13], [21], [23], [25], and [26]. In [12] and [16], it was noted that if we wish to avoid the prefiltering procedure, the multiwavelet basis should have extra approximation properties—that they be "K-balanced." For example, even though the GHM scaling functions $\phi_{i,0}$ are acceptable, the GHM multiwavelet filterbank must be preceded by a prefiltering stage. This is revealed by its balance order of zero or by graphing $\phi_{i,1}$. However, a high balance order by itself may not necessarily mean that the system will be suitable for signal processing applications either.

In this paper, examples of (symmetric) balanced orthogonal multiwavelet systems were given (in Figs. 3 and 4) for which the smoothness of the scaling functions depends on the chosen shift of $\underline{h}(n)$. Since shifting $\underline{h}(n)$ is equivalent to shifting the input $x(n)$, the behavior of the iterated filter bank is represented equally well by $\phi_{i,0}$ and by $\phi_{i,1}$. Therefore, although the balance order of a multiwavelet filter bank is the appropriate generalization of the approximation order of a scalar wavelet filter bank, it is informative to examine multiwavelet systems by looking at the expanded family of functions $\phi_{i,j}$. Assuming we want to avoid the use of prefilters in implementing the discrete multiwavelet transform, it is misleading to look at $\phi_{i,0}$ alone, whether or not the system is balanced. Regardless of the balance order, $\phi_{i,j}$ taken together reveals more fully the behavior of the iterated multiwavelet filterbank.

The minimal-length K-balanced orthogonal multiwavelet bases for $K = 1, 2, 3$ were presented and analyzed in light

TABLE I
THE COEFFICIENTS OF $g_i(n)$ DEFINING THE SCALING/WAVELET FILTERS/FUNCTIONS SHOWN IN Fig. 5.

$A = 13/80 - (1/40)\sqrt{151}$
$B = -(1/320)\sqrt{4720A + 2865}$
$C = (3/640)\sqrt{720A + 115}$
$g_0(0) = A/2 + 3/32$
$g_0(1) = -A$
$g_0(2) = -A/2 + 29/32$
$g_0(3) = A$
$g_1(0) = -9A/16 - 21/256$
$g_1(1) = 13A/16 + 29/256$
$g_1(2) = 5A/8 + 25/128$
$g_1(3) = -5A/8 + 115/128$
$g_1(4) = -A/16 - 29/256$
$g_1(5) = -3A/16 - 3/256$
$g_2(0) = 440AB/111 + 113B/222$
$g_2(1) = -1600AB/333 - 377B/333$
$g_2(2) = 40AB/3 - 29B/6$
$g_2(3) = B$
$g_3(0) = -40AC/7 - 11C/14$
$g_3(1) = 160AC/21 + 29C/21$
$g_3(2) = -160AC/21 + 230C/21$
$g_3(3) = -2080AC/21 + 1030C/21$
$g_3(4) = 40AC/3 - 29C/6$
$g_3(5) = C$

$$\begin{pmatrix} h_0(n) \\ h_1(n) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} g_0(0) & g_0(3) & g_0(1) & g_0(2) & g_0(2) & g_0(1) & g_0(3) & g_0(0) & 0 & 0 & 0 & 0 \\ g_1(0) & g_1(5) & g_1(1) & g_1(4) & g_1(2) & g_1(3) & g_1(3) & g_1(2) & g_1(4) & g_1(1) & g_1(5) & g_1(0) \end{pmatrix} \tag{5a}$$

of these issues. In particular, K-balanced multiwavelet bases based on even-length symmetric FIR filters were shown to be smoother than those based on odd-length symmetric FIR filters, as illustrated by the relations (4) and (5) and by the examples constructed (Fig. 5) via Gröbner basis techniques.

One of the challenges in extending these results to multiwavelets with higher multiplicity $(r > 2)$ is the nonlinear design equations that arise. The examples given in this paper appear to be at the edge of the practical limitations of Gröbner bases. For the investigation of multiwavelets with higher multiplicity the theory of multivariate resultants may be helpful [3].

The coefficients and the associated files for reproducing these results are available from the author or via the Internet at http://taco.poly.edu/selesi/.

APPENDIX A

A. Time-Reversal Relations

To prove (4) and (5), write

$$H_0(z) = \frac{1}{\sqrt{2}}\left(G_0(z^2) + z^{-(2L-1)}G_0(1/z^2)\right) \tag{6}$$

and

$$H_1(z) = \frac{1}{\sqrt{2}}\left(G_1(z^2) + z^{-(2L+3)}G_1(1/z^2)\right) \tag{7}$$

where

$$\begin{aligned} G_0(z) &= g_0(0) + \cdots + g_0(L-1)z^{-(L-1)} \\ G_1(z) &= g_1(0) + \cdots + g_1(L+1)z^{-(L+1)} \end{aligned}$$

giving

$$h_0(2n) = \frac{1}{\sqrt{2}}\, g_0(n) \tag{8}$$

$$h_0(2n+1) = \frac{1}{\sqrt{2}}\, g_0(L-1-n) \tag{9}$$

$$h_1(2n) = \frac{1}{\sqrt{2}}\, g_1(n) \tag{10}$$

and

$$h_1(2n+1) = \frac{1}{\sqrt{2}}\, g_1(L+1-n). \tag{11}$$

Exploiting the symmetry of h_0 and h_1, the dilation equation (1) can be written as

$$\begin{aligned} \phi_{0,0}(t) &= \sum_{n=0}^{L-1} g_0(n)(\phi_{0,0}(2t-n) \\ &\quad + \phi_{1,0}(2t+n-(L-1))) \\ \phi_{1,0}(t) &= \sum_{n=0}^{L+1} g_1(n)(\phi_{0,0}(2t-n) \\ &\quad + \phi_{1,0}(2t+n-(L+1))). \end{aligned}$$

The scaling functions generated by the shifted filters $h_i(n-1)$ are similarly given by

$$\begin{aligned} \phi_{0,1}(t) &= \sum_{n=0}^{L-1} g_0(n)(\phi_{0,1}(2t+n-L) + \phi_{1,1}(2t-n)) \\ \phi_{1,1}(t) &= \sum_{n=0}^{L+1} g_1(n)(\phi_{0,1}(2t+n-L-2) + \phi_{1,1}(2t-n)). \end{aligned}$$

Let $u_0(t) = \phi_{0,1}(L-t)$ and $u_1(t) = \phi_{1,1}(L+1-t)$. Then, with this substitution, we have $\phi_{0,1}(t) = u_0(L-t)$, $\phi_{1,1}(t) = u_1(L+1-t)$, and

$$\begin{aligned} u_0(L-t) = \sum_{n=0}^{L-1} g_0(n)(&u_0(L-2t-n+L) \\ &+ u_1(L+1-2t+n)) \end{aligned} \tag{12}$$

and

$$\begin{aligned} u_1(L+1-t) = \sum_{n=0}^{L+1} g_1(n)(&u_0(L-2t-n+L+2) \\ &+ u_1(L+1-2t+n)). \end{aligned} \tag{13}$$

Substituting $t = L - \alpha$ into (12) and $t = L + 1 - \alpha$ into (13) gives

$$\begin{aligned} u_0(\alpha) &= \sum_{n=0}^{L-1} g_0(n)(u_0(2\alpha-n) + u_1(2\alpha+n-(L-1))) \\ u_1(\alpha) &= \sum_{n=0}^{L+1} g_1(n)(u_0(2\alpha-n) + u_1(2\alpha+n-(L+1))). \end{aligned}$$

These are exactly the same dilation equations satisfied by $\phi_{i,0}$. Consequently, we obtain the relations (4) and (5). For multiwavelet bases with the structure (6) and (7), shifting $h_i(n)$ by a single sample has the effect of reflecting $\phi_0(t)$ about $t = L/2$ and reflecting $\phi_1(t)$ about $t = (L+1)/2$.

B. Polyphase CQF Property

It turns out that for even-length symmetric $h_i(n)$ to generate an orthogonal basis, the filters $g_i(n)$ must be conjugate quadrature filters (CQF's), that is, $g_i(n)$ must be orthogonal to their shifts by 2. This can be easily shown. For $i = j = 0$, the orthogonality condition (2) is

$$\sum_n h_0(n)h_0(n+4k) = \delta(k).$$

Splitting the left-hand side into two parts gives

$$\sum_n h_0(2n)h_0(2n+4k) + h_0(2n+1)h_0(2n+1+4k).$$

With (8) and (9), this becomes

$$\tfrac{1}{2}\sum_n (g_0(n)g_0(n+2k) + g_0(L-1-n)g_0(L-1-n-2k))$$

and therefore, we get

$$\sum_n g_0(n) g_0(n+2k) = \delta(k)$$

which is the well-known orthogonality condition for a two-channel orthogonal filter bank. The same result holds for $g_1(n)$.

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REFERENCES

[1] C. S. Burrus, R. A. Gopinath, and H. Guo, *Introduction to Wavelets and Wavelet Transforms*. Englewood Cliffs, NJ: Prentice-Hall, 1997.

[2] D. Cox, J. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. New York: Springer–Verlag, 1991.

[3] ——, *Using Algebraic Geometry*. New York: Springer–Verlag, 1998.

[4] G. C. Donovan, J. S. Geronimo, D. P. Hardin, and P. R. Massopust, "Construction of orthogonal wavelets using fractal interpolation functions.," *SIAM J. Math. Anal.*, vol. 27, no. 4, pp. 1158–1192, July 1996.

[5] T. R. Downie and B. W. Silverman, "The discrete multiple wavelet transform and thresholding methods," *IEEE Trans. Signal Processing*, vol. 46, pp. 3282–3291, Sep. 1998.

[6] J.-C. Faugére, F. M. de Saint-Martin, and F. Rouillier, "Design of regular nonseparable bidimensional wavelets using Gröbner basis techniques," *IEEE Trans. Signal Processing*, vol. 46, pp. 845–856, Jan. 1998.

[7] J. S. Geronimo, D. P. Hardin, and P. R. Massopust, "Fractal functions and wavelet expansions based on several scaling functions," *J. Approx. Theory*, vol. 78, no. 3, pp. 373–401, Sep. 1994.

[8] G.-M. Greuel, G. Pfister, and H. Schönemann, "Singular Reference Manual," in *Reports on Computer Algebra*. Kaiserslautern, Germany: Cent. Comput. Algebra, Univ. Kaiserslautern, May 1997, vol. 12.

[9] D. P. Hardin and D. W. Roach, "Multiwavelet prefilters I: Orthogonal prefilters preserving approximation order $p \leq 2$," *IEEE Trans. Circuits Syst. II*, vol. 45, pp. 1106–1112, Aug. 1998.

[10] Q. Jiang, "On the design of multifilter banks and orthonormal multiwavelet bases," *IEEE Trans. Signal Processing*, vol. 46, pp. 3293–3303, Dec. 1998.

[11] J. Lebrun and M. Vetterli, "Balanced multiwavelets theory and design.," *IEEE Trans. Signal Processing*, vol. 46, pp. 1119–1124, Apr. 1998.

[12] J. Lebrun and M. Vetterli, "High order balanced multiwavelets," *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process. (ICASSP)*, May 12–15, 1998.

[13] J. T. Miller and C.-C. Li, "Adaptive multiwavelet initialization," *IEEE Trans. Signal Processing*, vol. 46, pp. 3282–3291, Dec. 1998.

[14] H. Park, T. Kalker, and M. Vetterli, "Gröbner bases and multidimensional FIR multirate systems," *J. Multidimen. Syst. Signal Process.*, vol. 8, no. 4, pp. 11–30, 1997.

[15] P. Rieder and J. A. Nossek, "Smooth multiwavelets based on two scaling functions," in *Proc. IEEE Int. Symp. Time-Freq. Time-Scale Anal.*, Paris, France, June 18–21, 1996, pp. 309–312.

[16] I. W. Selesnick, "Multiwavelet bases with extra approximation properties," *IEEE Trans. Signal Processing*, vol. 46, pp. 2998–12 909, Nov. 1998.

[17] I. W. Selesnick, "Balanced GHM-like multiscaling functions," *IEEE Signal Processing Lett.*, vol. 6, pp. 111–112, May 1999.

[18] I. W. Selesnick and C. S. Burrus, "Maximally flat low-pass FIR filters with reduced delay," *IEEE Trans. Circuits Syst. II*, vol. 45, pp. 53–68, Jan. 1998.

[19] G. Strang and T. Nguyen, *Wavelets and Filter Banks*. Wellesley, MA: Wellesley–Cambridge, 1996.

[20] G. Strang and V. Strela, "Short wavelets and matrix dilation equations," *IEEE Trans. Signal Processing*, vol. 43, pp. 108–115, Jan. 1995.

[21] V. Strela, P. Heller, G. Strang, P. Topiwala, and C. Heil, "The application of multiwavelet filter banks to signal and image processing," *IEEE Trans. Image Processing*, vol. 8, pp. 548–563, Apr. 1999.

[22] J. Y. Tham, L.-X. Shen, S.-L. Lee, and H.-H. Tan, "A general approach for analysis and application of discrete multiwavelet transforms," , Preprint, 1997.

[23] M. J. Vrhel and A. Aldroubi, "Projection based prefiltering for multiwavelet transforms," *IEEE Trans. Signal Processing*, vol. 46, pp. 3088–3092, Nov. 1998.

[24] C. Weidmann, J. Lebrun, and M. Vetterli, "Significance tree image coding using balanced multiwavelets," in *Proc. Int. Conf. Image Process.*, Chicago, Oct. 1998.

[25] X.-G. Xia, "A new prefilter design for discrete multiwavelet transforms," *IEEE Trans. Signal Processing*, vol. 46, pp. 1558–1570, June 1998.

[26] X.-G. Xia, J. S. Geronimo, D. P. Hardin, and B. W. Suter, "Design of prefilters for discrete multiwavelet transforms," *IEEE Trans. Signal Processing*, vol. 44, pp. 25–35, Jan. 1996.

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