Optimization of Symmetric Self-Hilbertian Filters for the Dual-Tree Complex Wavelet Transform

Bogdan Dumitrescu, İlker Bayram, and Ivan W. Selesnick

***Abstract*—In this letter, we expand upon the method of Tay *et al.* for the design of orthonormal "Q-shift" filters for the dual-tree complex wavelet transform. The method of Tay *et al.* searches for good Hilbert-pairs in a one-parameter family of conjugate-quadrature filters that have one vanishing moment less than the Daubechies conjugate-quadrature filters (CQFs). In this letter, we compute feasible sets for one- and two-parameter families of CQFs by employing the trace parameterization of nonnegative trigonometric polynomials and semidefinite programming. This permits the design of CQF pairs that define complex wavelets that are more nearly analytic, yet still have a high number of vanishing moments.**

***Index Terms*—Complex wavelet, Hilbert pair, orthogonal filter banks, positive trigonometric polynomials.**

I. Introduction

The dual-tree complex wavelet transform (DT-$\mathbb{C}$WT), introduced by Kingsbury [8], is a useful extension of the conventional (real) wavelet transform (WT). Unlike the conventional wavelet transform, the DT-$\mathbb{C}$WT is nearly shift-invariant and is geometrically oriented in 2-D and higher dimensions. The one-dimensional DT-$\mathbb{C}$WT is comprised of two conventional (real) wavelet transforms operating in parallel on the input signal (making the DT-$\mathbb{C}$WT overcomplete by a factor of two). Therefore, the design of a DT-$\mathbb{C}$WT requires, in fact, the simultaneous design of two conventional wavelet transforms. More specifically, the DT-$\mathbb{C}$WT requires a pair of wavelet transforms where the respective wavelets form an approximate Hilbert-pair. That is, the wavelet transforms must approximately satisfy the condition

$$\psi^{\mathrm{d}}(t) = \mathcal{H}\{\psi(t)\} \tag{1}$$

where $\psi(t), \psi^{\mathrm{d}}(t)$ denote the respective wavelets, and where $\mathcal{H}\{\cdot\}$ denotes the Hilbert transform. Several methods for the design of such wavelet pairs have been proposed [13], [14] (see [11] for a review). In this letter, we expand upon the method proposed in [12].

As in [12], we consider orthonormal compactly-supported dyadic wavelet transforms only. That being the case, each wavelet transform is defined by a conjugate-quadrature filter (CQF). We denote by h_0 the low-pass CQF defining the first WT, and by h_0^{d} the low-pass CQF defining the second (dual) WT. The filters h_0 and h_0^{d} satisfy (1) if and only if they satisfy the "half-sample delay" condition as follows:

$$H_0^{\mathrm{d}}(e^{j\omega}) = e^{-j(0.5+b)\omega} H_0(e^{j\omega}), \quad |\omega| \le \pi \tag{2}$$

where $b \in \mathbb{Z}$. The "if" and "only if" parts were shown in [10] and [15], respectively, with $b = 0$. (See [3] for $b \in \mathbb{Z}$ and for the M-band case.) In this letter, as in [12], we consider $b = 0$ only. Because condition (2) cannot be satisfied exactly when both h_0 and h_0^{d} are FIR filters, the DT-$\mathbb{C}$WT calls for CQFs that satisfy (2) [and hence (1)] approximately.

One class of CQF pairs $\{h_0, h_0^{\mathrm{d}}\}$ for the DT-$\mathbb{C}$WT is the Q-shift solution introduced by Kingsbury [9] which is constrained to satisfy

$$h_0^{\mathrm{d}}(n) = h_0(N-n). \tag{3}$$

Consequently, the wavelets defined by h_0 and h_0^{d} satisfy $\psi^{\mathrm{d}}(t) = \psi(N-t)$. It can be shown that if a filter pair $\{h_0, h_0^{\mathrm{d}}\}$ satisfies (3) and approximately satisfies (2), then the phase of $H_0(e^{j\omega})$ will be approximately linear. However, the procedure in [9] produces wavelets having only one vanishing moment (VM). Recently, Tay *et al.* have proposed a new method for Q-shift pair design that produces wavelets having only one vanishing moment less than the maximum number possible, given the filter length [12].

The method [12] utilizes Bernstein polynomials which allows one to structurally impose the desired number of VM [2]. Allowing the number of VM to be one less than the maximum, a single degree of freedom becomes available for design. Consequently, a parameterization (using a single parameter) of the family of filters through the Bernstein polynomials is possible. The approach taken in [12] is to optimize the single free parameter so that (1) is satisfied as well as possible.

In this letter, we provide a complete characterization of the search space for the given VM conditions, showing that the use of Bernstein polynomials is restrictive. We provide parameterizations of this space for one and two degrees of freedom (by allowing the number of VM to be one and two less than the maximum, respectively).

II. Definitions

An orthonormal two-channel filter bank is completely determined by its low-pass filter $H_0(z)$. Let N be the degree of $H_0(z)$ and let $h_0(n)$ be real-valued. Denoting

$$P(z) = H_0(z)H_0(z^{-1}) = \sum_{k=-N}^{N} p_k z^{-k} \tag{4}$$

Manuscript received June 1, 2007; revised October 25, 2007. This work was supported by the Academy of Finland, under Project No. 213462, Finnish Centre of Excellence program (2006–2011). The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Gerald Matz.

B. Dumitrescu is with Tampere International Center for Signal Processing, Tampere University of Technology, FIN-33101 Tampere, Finland. He is on leave from "Politehnica" University of Bucharest, Bucharest, Romania (e-mail: bogdand@cs.tut.fi).

I. Bayram and I. W. Selesnick are with Department of Electrical and Computer Engineering, Polytechnic University, Brooklyn, NY 11201 USA (e-mail: bilker@gmail.com; selesi@poly.edu).

Digital Object Identifier 10.1109/LSP.2007.913609

the orthonormality condition is $P(z)+P(-z)=2$ and so the product filter (4) satisfies the Nyquist condition

$$p_{2k}=\delta_k. \tag{5}$$

The wavelet defined by h_0 has L vanishing moments if $H_0(z)$ has L roots at $z=-1$. On $P(z)$, this condition is written as

$$P(z)=R(z)(z^{-1}+2+z)^L \tag{6}$$

where

$$R(z)=\sum_{k=-N+L}^{N-L} r_k z^{-k}, \quad r_{-k}=r_k. \tag{7}$$

We note that, by construction, the symmetric polynomials $P(z)$ and $R(z)$ are nonnegative on the unit circle.

The low-pass CQF, h_0, can be obtained by performing spectral factorization on the product filter $P(z)$. The high-pass filter, h_1, is obtained from h_0 by $h_1(n)=(-1)^n h_0(N-n)$. The CQF set $\{h_0,h_1\}$ defines the wavelet $\psi(t)$ with L vanishing moments. For the DT-ℂWT, we require a second CQF set $\{h_0^{\mathrm{d}},h_1^{\mathrm{d}}\}$ which defines a wavelet $\psi^{\mathrm{d}}(t)$, such that the wavelet pair $\{\psi(t),\psi^{\mathrm{d}}(t)\}$ satisfies (1) approximately. As in [9] and [12], we ask in addition that the low-pass pair $\{h_0,h_0^{\mathrm{d}}\}$ satisfies (3). In this case, h_0 and h_0^{d} are spectral factors of the same filter $p(n)$. From [12], we have the following definition.

Definition 1: A **symmetric self-Hilbertian (SSH) filter** is a product filter with two spectral factors $\{h_0,h_0^{\mathrm{d}}\}$ which satisfy (3) and which define wavelets $\{\psi(t),\psi^{\mathrm{d}}(t)\}$ satisfying (1) approximately.

In order to measure how well a pair of wavelets $\{\psi(t),\psi^{\mathrm{d}}(t)\}$ satisfies (1), we use the "analyticity measures" defined in [12]. For this, let $\psi^c(t)$ denote the complex wavelet

$$\psi^c(t):=\psi(t)+j\psi^{\mathrm{d}}(t). \tag{8}$$

If the relation (1) were exact, then the wavelet $\psi^c(t)$ would be exactly analytic. On the other hand, if (1) holds approximately, then we expect $\psi^c(t)$ to be approximately analytic. This motivates the following definitions from [12]:

$$E_1=\frac{\max_{\omega<0}|\Psi^c(\omega)|}{\max_{\omega>0}|\Psi^c(\omega)|}$$
$$E_2=\frac{\int_{-\infty}^{0}|\Psi^c(\omega)|^2 d\omega}{\int_{0}^{\infty}|\Psi^c(\omega)|^2 d\omega}. \tag{9}$$

In the following, we investigate the positive polynomial family satisfying (5) and (6).

III. One Degree of Freedom

Given N, the degree of $H_0(z)$, the maximum number of vanishing moments is $(N+1)/2$, and the solutions are the Daubechies filters [4]. In this section, we set the number of vanishing moments to one less than the maximum, $L=(N-1)/2$. Then there are $L+1$ linear constraints (5) on the $L+2$ coefficients of the polynomial $R(z)$. So, there is a single degree of freedom in $R(z)$. We desire to parameterize, with a single parameter, all the polynomials (6) that are nonnegative and satisfy (5). A restrictive solution has been proposed in [12], using a particular form of Bernstein polynomials. Here we present a complete parameterization.

The polynomial (7) is nonnegative on the unit circle, and from (5) and (6), it satisfies the linear constraint

$$\boldsymbol{A}\boldsymbol{r}=\boldsymbol{e} \tag{10}$$

where $\boldsymbol{r}=[r_0\ r_1\ \ldots\ r_{N-L}]^T$ and $\boldsymbol{e}=[1\ 0\ \ldots\ 0]^T$. The size of $\boldsymbol{A}$ is $(L+1)\times(L+2)$. The set $\mathcal{S}\in\mathbb{R}^{L+2}$ of coefficients $\boldsymbol{r}$ satisfying these conditions is convex, as the intersection of the convex set of nonnegative polynomials of degree $N-L$ with the line defined by (10). Because the set of nonnegative Nyquist filters $P(z)$ is bounded, the set $\mathcal{S}$ is also bounded, and it is therefore a segment.

We choose r_0 as our parameter. We split $\boldsymbol{A}=[\boldsymbol{a}_0\tilde{\boldsymbol{A}}]$, where $\boldsymbol{a}_0$ is the first column of $\boldsymbol{A}$, and $\tilde{\boldsymbol{A}}$ is nonsingular ($\boldsymbol{A}$ has full row rank and any $L+1$ columns are linearly independent). Denoting $\tilde{\boldsymbol{r}}=[r_1\ \ldots\ r_{N-L}]^T$, we have

$$\tilde{\boldsymbol{r}}=\tilde{\boldsymbol{A}}^{-1}(\boldsymbol{e}-\boldsymbol{a}_0 r_0). \tag{11}$$

Hence, the other coefficients of $R(z)$ depend linearly on r_0.

Because the set $\mathcal{S}$ is a segment, the admissible values of r_0 form an interval $[r_{0,\min},r_{0,\max}]$. We can find the lower bound of the interval by solving the optimization problem

$$\begin{aligned} r_{0,\min}=\min_{\boldsymbol{r}\in\mathbb{R}^{L+2}}\quad & r_0\\ \text{s.t.}\quad & \boldsymbol{A}\boldsymbol{r}=\boldsymbol{e}\\ & R(\omega)\geq 0,\quad \forall\omega. \end{aligned} \tag{12}$$

Using the trace parameterization [1], [5]–[7] of nonnegative trigonometric polynomials, this becomes a semidefinite programming (SDP) problem and can be solved easily. The upper bound $r_{0,\max}$ is found by solving a maximization problem, otherwise identical to (12).

Comparison: We show that the proposed parameterization is better than that from [12] for the simple case $N=3$ (and $L=1$). Using the proposed approach, we obtain

$$P(z)=1+\frac{-2r_0+3}{4}(z^{-1}+z)+\frac{2r_0-1}{4}(z^{-3}+z^3) \tag{13}$$

with $r_0\in[3/8,3/2]$. Using Bernstein polynomials as in [12], we obtain

$$P(z)=1+\frac{-3\alpha+9}{16}(z^{-1}+z)+\frac{3\alpha-1}{16}(z^{-3}+z^3)$$

with $\alpha\in[0,0.5]$. Note that this corresponds to (13) with $r_0\in[3/8,9/16]$, i.e., only a subset of the admissible interval.

Optimization Procedure: To find the best self-Hilbertian filter $P(z)$, we follow a procedure similar to that of [12]. Since the measures (9) are nonconvex functions, with many local minima, we use an exhaustive search. We cover the interval $[r_{0,\min},r_{0,\max}]$ with a fine grid and, with the corresponding values r_0, generate product filters $R(z)$ using (11). We generate *all* the spectral factors of each $R(z)$ (in contrast to [12], where apparently a single approximately linear phase factor was used). Each spectral factor is multiplied with $(1+z^{-1})^L$ to obtain a candidate filter $H_0(z)$. The analyticity measures (9) are computed for each $H_0(z)$. Overall, the filter with smallest measure is retained.

Experimental Results: The parameters of the optimal filters obtained by the above procedure are shown in Table I. Almost all results are better than those reported in [12, Table I], with the

TABLE I
ANALYTICITY MEASURES OF E_1- AND E_2-OPTIMAL CQFS h_0 WITH ONE DEGREE OF FREEDOM. $r_{0,i}$ IS THE E_i-OPTIMAL r_0

N	L	$r_{0,1}$	$r_{0,2}$	$E_1(\%)$	$E_2(\%)$
3	1	0.37500	0.37505	10.23	1.60
5	2	0.24964	0.24816	10.27	1.44
7	3	0.08485	0.08470	2.27	0.044
9	4	0.10013	0.10017	6.24	0.404
11	5	0.05997	0.06006	3.82	0.170
13	6	0.05909	0.05908	3.64	0.135
15	7	0.04975	0.04971	4.04	0.135
17	8	0.03457	0.04137	4.09	0.142
19	9	0.02887	0.02901	5.24	0.196
21	10	0.02759	0.02761	2.90	0.066

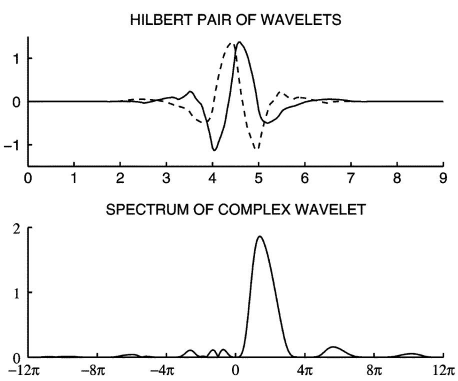

Fig. 1. Orthonormal Hilbert-pair with four vanishing moments, defined by E_1-optimal length-10 CQF. (Top) $\psi(t)$ and $\psi^{\mathrm{d}}(t)$. (Bottom) $|\Psi(\omega) + j\,\Psi^{\mathrm{d}}(\omega)|$.

exception of the values E_2 for $N = 5$ (where we have obtained a worse result, although our procedure should give at least the same optimal values) and $N = 11$ (same result as in [12]). For E_1, the improvement goes from only a few percent to values that are more than twice smaller, as when $N = 13$. For E_2, our values are typically two to three times smaller than those of [12]; for $N = 13$, our E_2 is more than ten times smaller. These improvements are due to checking all the spectral factors and not to the complete parameterization; the optimal values of r_0 are near $r_{0,\min}$, i.e., at the end of the interval covered by the parameterization of [12]. However, the new parameterization, besides being complete, helps in dealing with filters with several degrees of freedom, as in Section IV.

Example: The wavelets, $\psi(t)$ and $\psi^{\mathrm{d}}(t)$, defined by the length-10 E_1-optimal filter h_0, are illustrated in Fig. 1. The wavelets have four vanishing moments. In addition, the figure illustrates the magnitude of the Fourier transform of the complex wavelet, $|\Psi^c(\omega)| := |\Psi(\omega) + j\,\Psi^{\mathrm{d}}(\omega)|$. The coefficients $h_0(n)$ are tabulated in Table II under the heading [4 VM].

IV. TWO DEGREES OF FREEDOM

In this section, we set the number of vanishing moments to two less than the maximum. Therefore, we now use $L = (N-3)/2$. As a result, for a given degree N, we expect the complex wavelets obtained by optimizing over the two degrees of freedom will be more nearly analytic. There are $L+2$ linear constraints (5) on the $L+4$ coefficients of the polynomial $R(z)$ and therefore two degrees of freedom in the choice of $R(z)$. The set $\mathcal{S} \in \mathbb{R}^{L+4}$ of coefficients $\boldsymbol{r}$ is again convex and bounded and lies in the plane defined by (10).

TABLE II
COEFFICIENTS OF THE E_1-OPTIMAL h_0 DEFINING THE WAVELETS ILLUSTRATED IN FIGS. 1–3

	Length 10		Length 14
n	$h_0(n)$ [4 VM]	$h_0(n)$ [3 VM]	$h_0(n)$ [5 VM]
0	-0.0023380687	0.0001598067	0.0017293259
1	0.0327804569	0.0000007274	-0.0010305604
2	-0.0025090221	0.0235678740	-0.0128374477
3	-0.1187657989	0.0015148138	0.0018813576
4	0.2327030100	-0.0931304005	0.0359457035
5	0.7845762950	0.2161894746	-0.0395271550
6	0.5558782330	0.7761070855	-0.1048144141
7	0.0139812814	0.5778162235	0.2663807401
8	-0.0766273710	0.0004024156	0.7636351894
9	-0.0054654533	-0.0884144581	0.5651724402
10			0.0101286691
11			-0.1081211791
12			0.0133197551
13			0.0223511379

To parameterize $\boldsymbol{r}$, we may use any two coefficients r_i and r_ℓ, provided that the $(L+2)\times(L+2)$ matrix that results after removing the columns i and ℓ from $\boldsymbol{A}$ is nonsingular. For example, we can choose r_0 and r_1 as parameters. We split $\boldsymbol{A} = [\boldsymbol{a}_0\ \boldsymbol{a}_1\ \hat{\boldsymbol{A}}]$. Denoting $\hat{\boldsymbol{r}} = [r_2\ \ldots\ r_{N-L}]^T$, we have

$$\hat{\boldsymbol{r}} = \hat{\boldsymbol{A}}^{-1}(\boldsymbol{e} - \boldsymbol{a}_0 r_0 - \boldsymbol{a}_1 r_1). \tag{14}$$

Again, the other coefficients of $R(z)$ depend linearly on r_0 and r_1. (The only case where $\hat{\boldsymbol{A}}$ is singular is $N = 5$. In this case, the parameters can be r_0 and r_2.)

We show now how to cover the set of admissible r_0 and r_1 with a discrete set of points, aiming again at an exhaustive search for optimizing E_1 and E_2. We note that the solution of (12) and of the corresponding maximization problem still gives the interval $[r_{0,\min}, r_{0,\max}]$ of possible values for r_0 (note that this interval is larger than for a single degree of freedom). For a given $r_0 \in [r_{0,\min}, r_{0,\max}]$, the admissible values of r_1 lie in an interval $[r_{1,\min}(r_0), r_{1,\max}(r_0)]$, whose endpoints can be computed by solving

$$\begin{aligned} r_{1,\min}(r_0) = \min_{\boldsymbol{r}\in\mathbb{R}^{L+4}} \quad & r_1 \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{r} = \boldsymbol{e} \\ & r_0 = \text{fixed} \\ & R(\omega) \geq 0, \quad \forall\omega \end{aligned} \tag{15}$$

and the corresponding maximization problem. Again, the problem (15) can be expressed in SDP form.

Optimization Procedure: We take a grid covering $[r_{0,\min}, r_{0,\max}]$. For each r_0 on this grid, we find $r_{1,\min}(r_0), r_{1,\max}(r_0)$ by solving (15) and cover this interval with a grid of values r_1. For each r_0 and r_1, we compute $\boldsymbol{r}$ with (14) and then proceed as in the one freedom degree case. Practically, due to the high computation even for relatively small values of N, we restrain the search to values of r_0 near $r_{0,\min}$ (based on observations discussed in Section III). Also, we search first on a coarser grid (containing a few hundred r_0 points and 100 r_1 points) and then refine the search around the most promising points. The results we report are obtained after an overnight calculation for $N \leq 17$ and an over-week-end calculation for $N = 19$ and 21, on a standard PC. The

TABLE III
ANALYTICITY MEASURES OF E_1- AND E_2-OPTIMAL CQFs h_0 WITH TWO DEGREES OF FREEDOM. $r_{k,i}$ IS THE E_i-OPTIMAL r_k

N	L	$r_{0,1}$	$r_{1,1}$	$r_{0,2}$	$r_{1,2}$	$E_1(\%)$	$E_2(\%)$
5	1	0.38624	−0.05576	0.38365	−0.05704	1.68	0.101
7	2	0.17383	0.00377	0.13932	0.03223	1.48	0.0439
9	3	0.08612	−0.01625	0.08610	−0.01622	2.61	0.0473
11	4	0.11073	−0.07151	0.11233	−0.07285	3.28	0.0874
13	5	0.03758	−0.02311	0.03514	−0.02077	1.04	0.0163
15	6	0.02088	−0.01214	0.02083	−0.01210	2.24	0.0361
17	7	0.01300	−0.00799	0.01298	−0.00797	2.29	0.0357
19	8	0.01192	−0.00852	0.00972	−0.00651	3.48	0.105
21	9	0.00678	−0.00456	0.00685	−0.00462	2.70	0.0555

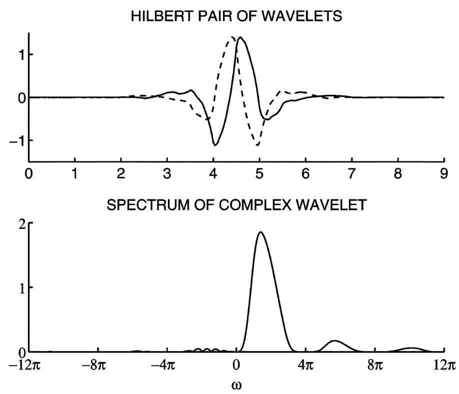

Fig. 2. Orthonormal Hilbert-pair with three vanishing moments, defined by E_1-optimal length-10 CQF. (Top) $\psi(t)$ and $\psi^{\mathrm{d}}(t)$. (Bottom) $|\Psi(\omega) + j\,\Psi^{\mathrm{d}}(\omega)|$.

complexity is dictated by the number of spectral factors, which grows exponentially, and by the computation of the analyticity measures E_1 and E_2; the SDP problems (12) and (15) have negligible impact on the total run-time.

Experimental results are shown in Table III. Due to the nature of the optimization problem, we are not certain that the results are globally optimal, especially for the larger values of N. However, they are clearly better than the results obtained using only one degree of freedom, especially for $N = 5, 9, 13$, where the improvement is substantial. The only case where practically no improvement was obtained is for E_2 and $N = 7$, which was already good.

Examples: The wavelets, $\psi(t)$ and $\psi^{\mathrm{d}}(t)$, defined by the length-10 E_1-optimal filter h_0 with two degrees of freedom are illustrated in Fig. 2. The coefficients of h_0 are tabulated in Table II under [3 VM]. The wavelets have three vanishing moments—one less than those illustrated in Fig. 1; however, the complex wavelet is more nearly analytic than the one of Fig. 1. We also illustrate the wavelets defined by the length-14 E_1-optimal filter in Fig. 3 (tabulated in Table II under [5 VM]). Again, the solution obtained using two degrees of freedom instead of one is substantially more nearly analytic.

V. CONCLUSION

In this letter, we have expanded upon the method of [12] for the design of orthonormal "Q-shift" filters for the dual-tree complex wavelet transform. The method of [12] searches for good Hilbert-pairs in a one-parameter family of conjugate-quadrature

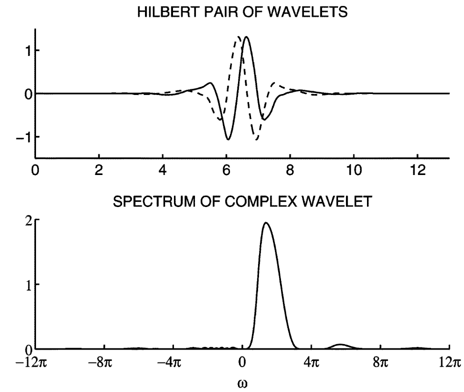

Fig. 3. Orthonormal Hilbert-pair with five vanishing moments, defined by E_1-optimal length-14 CQF. (Top) $\psi(t)$ and $\psi^{\mathrm{d}}(t)$. (Bottom) $|\Psi(\omega) + j\,\Psi^{\mathrm{d}}(\omega)|$.

filters that have one vanishing moment less than the Daubechies CQFs. In this letter, feasible sets are computed for one- and two-parameter families of CQFs by employing the trace parameterization of nonnegative trigonometric polynomials and semidefinite programming. This permits the design of CQF pairs that define complex wavelets that are more nearly analytic, yet still have a high number of vanishing moments.

REFERENCES

[1] B. Alkire and L. Vandenberghe, "Convex optimization problems involving finite autocorrelation sequences," *Math. Progr. Ser. A*, vol. 93, no. 3, pp. 331–359, Dec. 2002.

[2] H. Çağlar and A. N. Akansu, "A generalized parametric PR-QMF design technique based on Bernstein polynomial approximation," *IEEE Trans. Signal Process.*, vol. 41, no. 7, pp. 2314–2321, Jul. 1993.

[3] C. Chaux, L. Duval, and J.-C. Pesquet, "Image analysis using a dual-tree M-band wavelet transform," *IEEE Trans. Signal Process.*, vol. 15, no. 8, pp. 2397–2412, Aug. 2006.

[4] I. Daubechies*, Ten Lectures On Wavelets*. Philadelphia, PA: SIAM, 1992.

[5] B. Dumitrescu*, Positive Trigonometric Polynomials and Signal Processing Applications*. New York: Springer, 2007.

[6] B. Dumitrescu, I. Tăbuş, and P. Stoica, "On the parameterization of positive real sequences and MA parameter estimation," *IEEE Trans. Signal Process.*, vol. 49, no. 11, pp. 2630–2639, Nov. 2001.

[7] Y. Genin, Y. Hachez, Yu. Nesterov, and P. Van Dooren, "Optimization problems over positive pseudopolynomial matrices," *SIAM J. Matrix Anal. Appl.*, vol. 25, no. 1, pp. 57–79, Jan. 2003.

[8] N. G. Kingsbury, "Image processing with complex wavelets," *Phil. Trans. R. Soc. London A*, pp. 2543–2560, Sep. 1999.

[9] N. G. Kingsbury, "Design of Q-shift complex wavelets for image processing using frequency domain energy minimization," in *Proc. IEEE Int. Conf. Image Processing (ICIP)*, Sep. 2003.

[10] I. W. Selesnick, "Hilbert transform pairs of wavelet bases," *IEEE Signal Process. Lett.*, vol. 8, no. 6, pp. 170–173, Jun. 2001.

[11] I. W. Selesnick, R. G. Baraniuk, and N. G. Kingsbury, "The dual-tree complex wavelet transform—A coherent framework for multiscale signal and image processing," *IEEE Signal Process. Mag.*, vol. 22, no. 6, pp. 123–151, Nov. 2005.

[12] D. B. H. Tay, N. G. Kingsbury, and M. Palaniswami, "Orthonormal Hilbert-pair of wavelets with (almost) maximum vanishing moments," *IEEE Signal Process. Lett.*, vol. 13, no. 9, pp. 533–536, Sep. 2006.

[13] I. Voicu and M. Borda, "New method of filters design for dual tree complex wavelet transform," in *Proc. Int. Symp. Signals, Circuits, and Systems*, Jul. 2005, vol. 2, pp. 437–440.

[14] R. Yu, "Characterization and sampled-data design of dual-tree filter banks for Hilbert transform pairs of wavelet bases," *IEEE Trans. Signal Process.*, vol. 55, no. 6, pp. 2458–2471, Jun. 2007.

[15] R. Yu and H. Özkaramanlı, "Hilbert transform pairs of orthogonal wavelet bases: Necessary and sufficient conditions," *IEEE Trans. Signal Process.*, vol. 53, no. 12, pp. 4723–4725, Dec. 2005.