Interpolating Multiwavelet Bases and the Sampling Theorem

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***Abstract*—This paper considers the classical sampling theorem in multiresolution spaces with scaling functions as interpolants. As discussed by Xia and Zhang, for an orthogonal scaling function to support such a sampling theorem, the scaling function must be cardinal (interpolating). They also showed that the only orthogonal scaling function that is both cardinal and of compact support is the Haar function, which is not continuous. This paper addresses the same question, but in the *multi*wavelet context, where the situation is different. This paper presents the construction of compactly supported orthogonal multiscaling functions that are continuously differentiable and cardinal. The scaling functions thereby support a Shannon-like sampling theorem. Such wavelet bases are appealing because the initialization of the discrete wavelet transform (prefiltering) is the identity operator.**

***Index Terms*— Filter banks, multiwavelet bases, sampling, wavelet transforms.**

I. Introduction

SHANNON'S sampling theorem for bandlimited signals is one of the cornerstones of signal processing and communication theory. Indeed, the representation of a function by its samples is an important question with a long history. While the Shannon sampling theorem is based on bandlimited signals, it is natural to investigate other signal classes for which a sampling theorem holds. The assumption that a signal is bandlimited, although eminently useful, is not always realistic. Note that i) bandlimited signals are of infinite duration, and ii) the sinc function, which is used to reconstruct a bandlimited function from its samples, is of infinite support and decays only as $|1/x|$. We are particularly interested in sampling theorems for signals of finite duration and for which the reconstruction function is also of compact support.

To this end, note that the sinc function is one of the primary examples of an orthogonal scaling function from the theory of wavelet bases. The sinc function generates a scaling space V in the context of multiresolution analysis and serves as the interpolant in the context of the sampling theorem. The question naturally arises—are there orthogonal wavelet bases for which the scaling function both i) supports a sampling theorem in the same fashion and ii) is of compact support? (After all, orthogonal wavelets gained importance with the construction of scaling functions having compact support in [9].) Unfortunately, the Haar scaling function is the only orthogonal scaling function of compact support for which a Shannon-like sampling property holds, as proven in [39].

This paper takes up the same question but in the context of *multi*wavelet bases (wavelet bases based on more than a single scaling function), where the situation is different. This paper shows, via the construction of examples, that for orthogonal multiwavelet bases, it is possible for the scaling functions to achieve simultaneously the sampling property, compact support, and approximation order $K > 1$.

A variety of results regarding wavelet bases and sampling theorems have already been described. Walter has given a sampling theorem describing the reconstruction of a function f in a scaling space V from its samples [35]. Walter's theorem does not require that the scaling function $\phi(t)$ be cardinal (interpolatory, see below); however, the interpolant is generally not the same function as the scaling function and is generally of infinite support. Aldroubi and Unser have considered wavelet sampling and the role of cardinal scaling functions, especially in the context of biorthogonal bases [1]–[3], [34]. Cardinal scaling functions (nonorthogonal) are also discussed in [5], [11], and [33]. The notion of scale-limited signals and the issue of translation invariance in wavelet sampling is discussed in [14]; see also [4]. Systems for the reconstruction of nonbandlimited (finite duration) signals from samples, and the implementation of such systems, is considered in [17]. Other recent results regarding the properties of multiwavelet bases in interpolation are in [21]. In [38], Xia and Suter discuss vector-valued wavelets and describe how the interpolation property of a vector-valued scaling function $\underline{\phi}(t)$ is reflected in the structure of the vector-valued scaling filter $\underline{h}(n)$. The examples constructed in Section V conform to the structure described in [36, Prop. 5] and have, in addition, a maximal number of (balanced) zero moments given their supports.

II. Preliminaries

From the classical Shannon sampling theorem, if $f(t)$ is bandlimited to $(-\pi, \pi)$, then

$$f(t) = \sum_n f(n)\,\mathrm{sinc}\,(t-n)$$

where

$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

Throughout the paper, t is real, and n is integer.

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An important property of the sinc function is that it is a cardinal function. A function $\phi(t)$ is said to be a *cardinal function* if

$$\phi(n) = \delta(n) = \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n = \pm 1, \pm 2, \cdots. \end{cases}$$

Cardinal functions take the value 0 on the nonzero integers and take the value 1 at $t = 0$.

From the theory of wavelet bases, a function $\phi(t)$ is said to be an *orthogonal scaling function* if

1) $\phi(t)$ satisfies a dilation equation

$$\phi(t) = \sqrt{2} \sum_n h(n)\phi(2t - n)$$

where $h(n)$ is known as the scaling filter.

2) $\phi(t)$ is orthogonal to its integer shifts

$$\int \phi(t)\phi(t - n)\, dt = \delta(n).$$

The sinc function *is* such a function, with the additional property that it is cardinal [35]. The sinc function is a *cardinal orthogonal scaling function*, or COSF.

From the theory of wavelet bases, the *scaling space* $V_j(\phi)$ associated with a scaling function $\phi(t)$ is

$$V_j(\phi) = \text{Span}_n\, \{\phi(2^j t - n)\}.$$

The Shannon sampling theorem can then be stated in the wavelet context. Let $\phi(t)$ be the sinc function. If $f(t) \in V_0(\phi)$, then

$$f(t) = \sum_n f(n)\phi(t - n).$$

Does the sampling theorem hold for other $\phi(t)$? It is shown in [39] that the sampling property holds for an orthogonal scaling function $\phi(t)$ if and only if $\phi(t)$ is also cardinal. Therefore, every cardinal orthogonal scaling function yields a sampling theorem. The Shannon sampling theorem for bandlimited signals is the special case obtained using the sinc function.

An important question arises—do there exist cardinal orthogonal scaling functions of compact support? The answer is yes: The Haar function is one example. However, as mentioned in the Introduction, there are no others [39]. The orthogonal scaling functions of Daubechies, for example, are not cardinal. The Haar function is the only cardinal orthogonal scaling function of compact support.

Halfband Filters

It is convenient to characterize a compactly supported COSF in terms of the scaling filter h. Recall first the definition of a halfband filter. $h(n)$ is *halfband* if $h(2n) = c \cdot \delta(n)$ for some nonzero c. Halfband filters take the value 0 on the even integers, except at $n = 0$. The properties and design of halfband filters are summarized in detail in [26].

Let $r(n)$ denote the autocorrelation sequence of $h(n)$. $r(n) = \Sigma_k\ h(k)h(k + n)$. For $h(n)$ to generate an *orthogonal* scaling function $\phi(t)$, it is necessary that $r(n)$ be halfband $r(2n) = \delta(n)$ [10]. On the other hand, for a scaling filter $h(n)$ to generate a *cardinal* scaling function $\phi(t)$, it is necessary that $h(n)$ be halfband, $h(2n) = (1/\sqrt{2})\delta(n)$ [39]. Hence, for $h(n)$ to generate a scaling function that is both cardinal *and* orthogonal, it is necessary that both $h(n)$ and $r(n)$ be halfband.

The Haar function is the only COSF of compact support because the only appropriate FIR halfband filters $h(n)$ whose autocorrelation function are also halfband are filters having two nonzero coefficients. Examples of noncompactly supported COSF's, given in [27] and [39], are based on IIR scaling filters $h(n)$. Although they are not of compact support, their decay is exponential. Note that scaling functions based on scaling filters of the form $H(z) = A(z^2) + z^{-(2d+1)}$, where $A(z)$ is allpass, are cardinal because such scaling filters are halfband (up to a shift). The scaling function in [27, Fig. 2] is therefore a COSF, as it is based on a filter of that type. Examples of IIR COSF's, based on nonrational transfer functions, have been recently given in [36] and [37].

III. Multiwavelet Bases and the Sampling Theorem

Multiwavelet bases have received much attention since 1994 when it was shown by example in [12], [13], and [32] that symmetry, orthogonality, compact support, and approximation order $K > 1$ can be simultaneously achieved, which is not possible in the traditional scalar wavelet case.[1]

In this paper, we show by using multiwavelet bases that it is possible to achieve simultaneously cardinality, orthogonality, compact support, and approximation order $K > 1$. That is, there exist orthogonal multiwavelet scaling functions of compact support and approximation order $K > 1$ for which a Shannon-like sampling property holds, which is not possible in the scalar wavelet case.

Multiwavelet bases are wavelet bases based on several scaling and wavelet functions. This paper considers multiwavelet bases based on two scaling functions $\phi_0(t)$ and $\phi_1(t)$ and two wavelet functions $\psi_0(t)$ and $\psi_1(t)$. Accordingly, there are two scaling filters $h_0(n)$ and $h_1(n)$ and two wavelet filters $h_2(n)$ and $h_3(n)$.

The functions $\phi_0(t), \phi_1(t)$ are *orthogonal multiscaling functions* if

1) $\phi_0(t)$ and $\phi_1(t)$ satisfy a *matrix* dilation equation

$$\underline{\phi}(t) = \sqrt{2} \sum_n C(n)\underline{\phi}(2t - n) \tag{1}$$

where $\underline{\phi}(t) = (\phi_0(t), \phi_1(t))^t$, and $C(n)$ are 2×2 matrices.

2) $\phi_0(t)$ and $\phi_1(t)$ are orthogonal to their integer shifts.

$$\int \phi_i(t)\phi_j(t - n)\, dt = \delta(i - j) \cdot \delta(n).$$

The notation for $C(n)$ used in this paper is $[C(n)]_{i,j} = h_i(2n + j)$. For example

$$C(0) = \begin{pmatrix} h_0(0) & h_0(1) \\ h_1(0) & h_1(1) \end{pmatrix}, \quad C(1) = \begin{pmatrix} h_0(2) & h_0(3) \\ h_1(2) & h_1(3) \end{pmatrix}$$

etc., where $h_0(n)$ and $h_1(n)$ are the two scaling filters.

[1] To distinguish multiwavelet bases from wavelet bases based on a single scaling function, we will call the later *scalar* wavelet bases.

The scaling space $V_j(\phi_0, \phi_1)$ is given by

$$V_j(\phi_0, \phi_1) = \text{Span}_n \{\phi_0(2^j t - n), \phi_1(2^j t - n)\}.$$

The functions $\phi_0(t)$ and $\phi_1(t)$ will be called *cardinal* if

$$\phi_0(n/2) = \delta(n)$$
$$\phi_1(n/2) = \delta(n-1).$$

Except for $t = 0$, $\phi_0(t)$ takes the value 0 on the half integers, and except for $t = \frac{1}{2}$, so does $\phi_1(t)$.

A version of the sampling theorem, for the multiwavelet case, is straightforward. Let $\phi_0(t)$ and $\phi_1(t)$ be cardinal orthogonal multiscaling functions. If $f(t) \in V_0(\phi_0, \phi_1)$, then

$$f(t) = \sum_n f(n)\phi_0(t-n) + f(n+1/2)\phi_1(t-n).$$

Shannon sampling using the sinc function can be expressed in this form using $\phi_0(t) = \text{sinc}(2t), \phi_1(t) = \text{sinc}(2t-1)$.

The question becomes: do there exist cardinal orthogonal *multi*scaling functions $\phi_0(t)$ and $\phi_1(t)$ of compact support and approximation order $K > 1$? Yes. In Section V, examples of such functions will be given.

IV. Balance Order

For scalar wavelet bases, the number of zero moments is an important measure of how well the discrete-time wavelet transform (DWT) compresses smooth signals. Recall that scalar wavelet bases with K zero moments[2] are characterized by the well-known condition that $H_0(z)$ must have a K-order zero at $z = -1$. For multiwavelets with K zero moments,[3] this condition has been generalized; see, for example, [7] and [16].

Nevertheless, it is important to highlight a difference between scalar- and multiwavelet bases. For scalar wavelet bases, the associated filter bank inherits the zero moment properties of the basis. However, in the multiwavelet case, the situation is different. For multiwavelet bases, the associated filter bank does *not* necessarily inherit the zero moment properties of the basis [19], [28].

To be specific, for scalar wavelet bases with K zero moments, the lowpass/highpass channels of the associated filter bank preserve/annihilate the set $\mathcal{P}_{K-1}$ of polynomials of degree $k < K$; see, for example, [30, Th. 1]. However, in the multiwavelet case considered here, to guarantee the preservation/annihilation properties of the associated filter bank, it is not sufficient that the multiwavelet basis have K zero moments. A stronger condition is required. Multiwavelet bases for which the zero moment properties *do* carry over to the discrete-time filter bank are called *balanced* after Lebrun and Vetterli [18], [19].

Specifically, multiwavelet bases for which the associated filter bank preserves/annihilates the set $\mathcal{P}_{K-1}$ of polynomials of degree $k < K$ are said to be *order-K balanced*. See [18], [19], [28], and [29] for further details. Filter banks based on unbalanced multiwavelet bases require specialized prefilters.

From [28], the condition for order-1 balancing for multiwavelet bases is

$$(z^{-3} + z^{-2} + z^{-1} + 1) \quad \text{divides} \quad H_0(z) + H_1(z).$$

Order-1 balanced multiwavelet filter banks preserve/annihilate constant signals. From [28], the condition for order-2 balancing is

$$(z^{-3}+z^{-2}+z^{-1}+1)^2 \text{ divides } H_0(z)+\left(\frac{3-z^{-4}}{2}\right)H_1(z). \quad (2)$$

Order-2 balanced multiwavelet filter banks preserve/annihilate ramp and constant signals. A condition of this type for order-K balancing, for general K, is introduced in [28]. An equivalent condition for K-balancing is given in [19] and [20]. The examples to be given in Section V will be balanced up to their approximation order.

V. Cardinal Multiwavelet Bases

To obtain cardinal orthogonal multiscaling functions, it is useful to characterize them in terms of the scaling filters h_0 and h_1. For h_0, h_1 to generate orthogonal scaling functions ϕ_0, ϕ_1, it is necessary that h_0 and h_1 be orthogonal to their shifts by 4. Specifically

$$\sum_n h_i(n)h_j(n+4k) = \delta(i-j) \cdot \delta(k). \quad (3)$$

This is the condition that characterizes the orthogonality of four-channel filter banks. It arises here because the two-channel multiwavelet filter bank can be drawn as a four-channel *scalar* filter bank with interleaving of subband signals [25].

The scaling functions ϕ_0 and ϕ_1 presented below are based on scaling filters h_0 and h_1 possessing a particular structure $H_0(z) = z^{-n} + G_0(z^2)$ and $H_1(z) = z^{-(n+2)} + G_1(z^2)$. From the dilation equation (1), it is quite direct that this structure generates cardinal scaling functions, as noted in [38].

A. Order-2 Balanced Example

We obtained an order-2 balanced cardinal orthogonal system with scaling functions supported on [0,5] and scaling filters of length 11. The scaling filters have the form

$$h_0(n) = \frac{1}{\sqrt{2}} \left(a, 0, b, 1, \quad c, 0, d, 0, \quad e, 0, f \right) \quad (4)$$

$$h_1(n) = \frac{1}{\sqrt{2}} \left(-f, 0, e, 0, -d, 1, c, 0, -b, 0, a\right) \quad (5)$$

for $n = 0, \cdots, 10$. This structure can also be written in terms of $H_0(z)$ and $H_1(z)$ using their polyphase decompositions

$$H_0(z) = \frac{1}{\sqrt{2}} \left(z^{-3} + G(z^2)\right)$$

$$H_1(z) = \frac{1}{\sqrt{2}} \left(z^{-5} + z^{-10}G(-z^{-2})\right)$$

where $G(z) = a + bz^{-1} + \cdots + fz^{-5}$. With this form, orthogonality between h_0 and h_1 is structurally incorporated.

[2] If $\int t^k \psi(t)\, dt = 0$ for $k = 0, \cdots, K-1$ (and not for $k = K$), then the wavelet basis is said to have approximation order K or K zero moments.

[3] If $\int t^k \psi_i(t)\, dt = 0$ for $i = 0, 1$ and $k = 0, \cdots, K-1$ (and not for $k = K$), then the multiwavelet basis is said to have approximation order K or K zero moments.

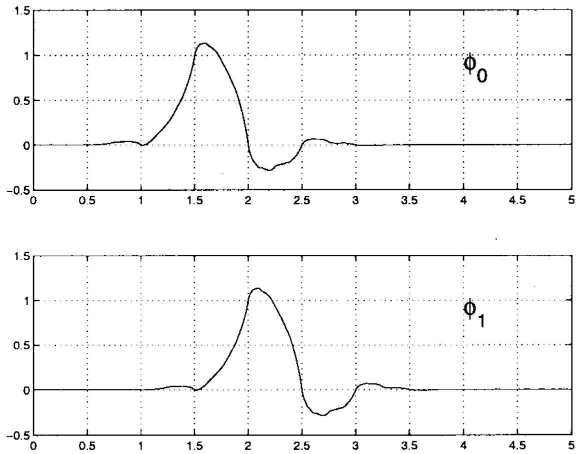

Fig. 1. Order-2 balanced cardinal orthogonal scaling functions $\phi_0(t)$ and $\phi_1(t)$, with $A = -1/8 + \sqrt{15}/32$. The support of each is [0, 5].

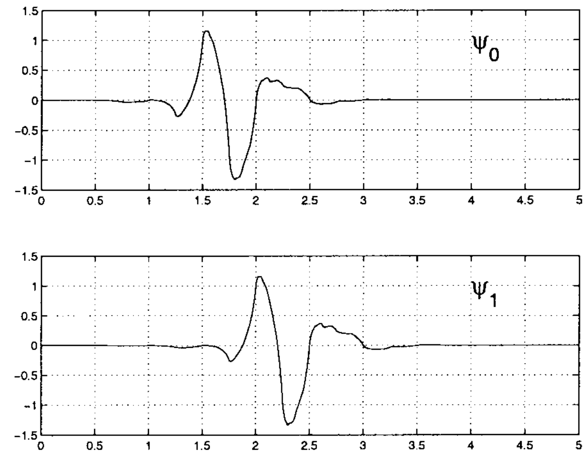

Fig. 2. Order-2 balanced wavelet functions $\psi_0(t)$ and $\psi_1(t)$, corresponding to the scaling functions shown in Fig. 1. Like ϕ_0, ϕ_1, the wavelets ψ_0, ψ_1 are cardinal.

For any values $a, \cdots, f$, the filter h_0 is orthogonal to h_1 and to shifts of h_1 by 4. In addition, if h_0 is orthogonal to its own shifts by 4, then so is h_1. It is therefore necessary only to choose the filter parameters so that h_0 is orthogonal to its own shifts by 4. The remaining free parameters will be used to attain balance order $K > 1$.

Our problem is to find $a, \cdots, f$ such that h_0 and h_1 in (4) and (5) satisfy the orthogonality conditions (3) and the second order balancing conditions (2). This is a system of nonlinear equations—the balancing conditions (2) are linear, but the orthogonality conditions (3) are quadratic. The following solutions to this system of nonlinear equations were obtained using a Göbner basis [8] (for the computation of which, the software *Singular* was employed [15]).

$$
\begin{aligned}
A &= -1/8 \pm \sqrt{15}/32 \\
a &= 1/32 \\
b &= A + 1/4 \\
c &= 15/16 \\
d &= -2A - 1/4 \\
e &= 1/32 \\
f &= A.
\end{aligned}
$$

As indicated, two solutions exist; however, only one of them yields acceptable scaling functions, namely, $A = -1/8 + \sqrt{15}/32$. That solution is shown in Fig. 1. The Sobolev exponent was found to be no less than 1.526. It follows that ϕ_0 and ϕ_1 are continuous and differentiable. Note that ϕ_0 and ϕ_1 shown in the figure are actually shifted cardinal functions $\phi_0(3/2) = 1$ instead of $\phi_0(0) = 1$, etc. The other solution, corresponding to $A = -1/8 - \sqrt{15}/32$, generates scaling functions that appear fractal.

According to [6], it was shown in [22] that if neither $C(0)$ nor $C(M)$ is nilpotent, then the support[4] of $\underline{\phi}(t)$ is $[0, M]$. Here, $C(M)$ is the highest coefficient in (1). Therefore, the scaling functions in Fig. 1 have, together, support [0, 5]. It was verified numerically that ϕ_0 and ϕ_1 each have support [0, 5].

[4] supp $\underline{\phi}$ = supp ϕ_0 $\cup$ supp ϕ_1

The scaling functions in Fig. 1, generated by h_0, h_1, look similar to each other and to the noncompactly supported cardinal orthogonal scaling function in [39, Figs. 1(a) and 3(a)]. However, in [39], the single scaling function $\phi(t)$ is based on a single IIR filter, whereas here, the two scaling functions $\phi_0(t)$ and $\phi_1(t)$ are based on two (but related) FIR filters h_0 and h_1.

The wavelet filters h_2 and $h_3$are given by

$$h_2(n) = (-a, 0, -b, 1, -c, 0, -d, 0, -e, 0, -f) \quad (6)$$
$$h_3(n) = (\ \ f, 0, -e, 0,\ \ d, 1, -c, 0,\ \ \ b, 0, -a) \quad (7)$$

for $n = 0, \cdots, 10$ or, equivalently, in terms of polyphase components

$$
\begin{aligned}
H_2(z) &= \frac{1}{\sqrt{2}}(z^{-3} - G(z^2)) \\
H_3(z) &= \frac{1}{\sqrt{2}}(z^{-5} - z^{-10}G(-z^{-2})).
\end{aligned}
$$

Note that h_2 and h_3 use the same values $a, \cdots, f$. All four analysis filters are obtained from the single prototype filter h_0. The special structure for h_0, h_1, h_2, h_3 guarantees orthogonality (3), provided that h_0 is orthogonal to its shifts by 4. In addiiton, note that $H_2(z) = -H_0(-z)$ and that $H_3(z) = -H_1(-z)$.

The use of a structure guaranteeing orthogonality was also used in [23] and [25], where the structure employed gave rise not to cardinal functions but to symmetric (unbalanced) ones.

The wavelets $\psi_0(t)$ and $\psi_1(t)$ shown in Fig. 2 are given by

$$\underline{\psi}(t) = \sqrt{2} \sum_n D(n)\underline{\phi}(2t - n)$$

where

$$D(0) = \begin{pmatrix} h_2(0) & h_2(1) \\ h_3(0) & h_3(1) \end{pmatrix}, \quad D(1) = \begin{pmatrix} h_2(2) & h_2(3) \\ h_3(2) & h_3(3) \end{pmatrix}$$

etc.

1) Zeros at π: It turns out that the lowpass filters h_0 and h_1 each posses a double zero at $\omega = \pi$. This condition characterizes scalar wavelets with two zero moments, but for multiwavelets (both unbalanced and balanced), it is, in general, neither typical nor necessary. The reason h_0 and h_1 each have a double zero at $\omega = \pi$ in this multiwavelet example turns out to be straightforward. For a multiwavelet filter bank to be order-K balanced, the highpass channels must annihilate discrete-time polynomials of degree $k < K$. The annihilation of $\mathcal{P}_{K-1}$ by the highpass filters h_2 and h_3 implies, in turn, that $(z-1)^K$ divides both $H_2(z)$ and $H_3(z)$. Because $H_2(z) = -H_0(-z)$ and $H_3(z) = -H_1(-z)$, it follows that $(z+1)^K$ divides both $H_0(z)$ and $H_1(z)$. That is, the lowpass filters h_0 and h_1 each posses a K-order zero at $\omega = \pi$.

2) Polyphase CQF Property: It turns out that the polyphase component $G(z)$ must be a conjugate quadrature filter (CQF), that is, $g(n)$ must be orthogonal to its shifts by 2. To show this, begin with the orthogonality condition (3) for $i = j = 0$

$$\sum_n h_0(n)h_0(n+4k) = \delta(k)$$

and split the left-hand side into two parts

$$\sum_n h_0(2n)h_0(2n+4k) + \sum_n h_0(2n+1)h_0(2n+1+4k).$$

Note that $h_0(n) = g(n)/\sqrt{2}$ and $h_0(2n+1) = g(n-1)\sqrt{2}$. We then get

$$\tfrac{1}{2}\sum_n g(n)g(n+2k) + \tfrac{1}{2}\sum_n \delta(n-1)\delta(n+2k-1) = \delta(k)$$

or simply

$$\sum_n g(n)g(n+2k) = \delta(k)$$

which is the classical orthogonality condition for a two-channel orthogonal system. Consequently, the efficient methods for implementing and parameterizing such systems in terms of lattice angles (see [24], for example) can be immediately utilized for the multiwavelet systems described in this paper. (We clarify, however, that all computations in this paper use the coefficients $h_i(n)$.)

B. Order-3 Balanced Example

The same procedure was used to design an order-3 balanced cardinal orthogonal multiwavelet system. In this case, the filters h_i are of length 15 and are given by

$$H_0(z) = \frac{1}{\sqrt{2}}(z^{-5} + G(z^2))$$
$$H_1(z) = \frac{1}{\sqrt{2}}(z^{-7} - z^{-14}G(-z^{-2}))$$

where $G(z) = g(0) + g(1)z^{-1} + \cdots + g(7)z^{-7}$. From [28], the condition for order-3 balancing is

$$(z^{-3} + z^{-2} + z^{-1} + 1)^3 \quad \text{divides}$$

$$H_0(z) + \left(\frac{15 - 10z^{-4} + 3z^{-8}}{8}\right)H_1(z). \tag{8}$$

The solutions for $G(z)$ such that the scaling filters h_0 and h_1 satisfy the orthogonality (3) and third order balancing (8) conditions are

$$\begin{aligned}
A &= 3/1280 \pm \sqrt{31}/2560\\
g(0) &= 2A + 1/512\\
g(1) &= -A + 1/32\\
g(2) &= -6A + 125/512\\
g(3) &= 3A + 15/16\\
g(4) &= 6A - 125/512\\
g(5) &= -3A + 1/32\\
g(6) &= -2A - 1/512\\
g(7) &= A.
\end{aligned}$$

The value $A = 3/1280 + \sqrt{31}/2560$ generates the smoother scaling functions. Surprisingly, these three-balanced multiscaling functions ϕ_0 and ϕ_1 resemble the two-balanced scaling functions shown in Fig. 1 so closely that they are almost indistinguishable. Moreover, these scaling functions are even more similar to one another than are the two-balanced ones in Fig. 1. In addition, they do not appear to be any smoother than the two-balanced solution. The Sobolev exponent was found to be no less than 1.335.

C. Order-4 Balanced Example

We were also able to obtain a four-balanced cardinal orthogonal multiwavelet system with filters h_i of length 23.

$$H_0(z) = \frac{1}{\sqrt{2}}(z^{-9} + G(z^2))$$
$$H_1(z) = \frac{1}{\sqrt{2}}(z^{-11} - z^{-22}G(-z^{-2}))$$

where $G(z) = g(0) + g(1)z^{-1} + \cdots + g(11)z^{-11}$. From [28], the condition for order-4 balancing is

$$(z^{-3} + z^{-2} + z^{-1} + 1)^4 \quad \text{divides}$$

$$H_0(z) + \left(\frac{35 - 35z^{-4} + 21z^{-8} - 5z^{-12}}{16}\right)H_1(z). \tag{9}$$

The solutions for $G(z)$ such h_0 and h_1 satisfy orthogonality (3) and fourth-order balancing (9) conditions are

$$
\begin{aligned}
A &= -67/40960 \pm \sqrt{17951}/81920\\
B &= -2A + 7/16\,384\\
&\quad \pm \sqrt{6\,710\,886\,400\,A^2 - 4\,259\,840\,A + 305}/40\,960\\
g(0) &= B + 4\,A - 7/8192\\
g(1) &= -A - 5/2048\\
g(2) &= -3\,B - 16\,A - 9/2048\\
g(3) &= 5\,A + 21/512\\
g(4) &= 2\,B + 24\,A + 1099/4096\\
g(5) &= -10\,A + 945/1024\\
g(6) &= 2\,B - 16\,A - 553/2048\\
g(7) &= 10\,A + 21/512\\
g(8) &= -3\,B + 4\,A + 57/8192\\
g(9) &= -5\,A - 5/2048\\
g(10) &= B\\
g(11) &= A.
\end{aligned}
$$

In this case, four solutions emerge. The smoothest pair of scaling functions are more similar to one another and only slightly smoother than the two-balanced example. Their Sobolev exponent was found to be no less than 1.798. Again, like the three-balanced solution, this four-balanced solution closely resembles the two-balanced solution in Figs. 1 and 2.

D. Symmetry

We also investigated the design of examples that have, in addition, symmetry properties; however, we were not able to find any such examples, unfortunately, even for higher multiplicity multiwavelets.

VI. Discussion

A. Smoothness

It is disappointing that as the balance order is increased, the cardinal multiscaling functions do not become significantly smoother. Apparently, the interpolation property, taken together with orthogonality and compact support, is quite restrictive. We suppose that by increasing the multiplicity of the multiwavelet basis (the number of scaling and wavelet functions) to $r > 2$, smoother solutions might be available.

B. Scaling Function Similarity

As noted by Rieder and Nossek in [25], the two-channel multiwavelet filter bank can be drawn as a four-channel *scalar* filter bank with interleaving of subband signals. See also [18] and [28]. If the filters are too different, then the interleaving becomes a problem when the filter bank is iterated on one of its subband signals. Either a prefilter is required, or the filters must be appropriately designed. Certainly, when one scaling filter is simply the shift of the other (or nearly so), then the interleaving of subband signals presents no problem. For the multiscaling functions considered in this paper, balancing conditions appear to make them similar to one another, with higher balancing leading to greater similarity. A similar phenomenon occurs for the multiscaling functions presented in [19]; as the balance order increases, the two scaling functions resemble each other more.

C. Prefiltering

The use of cardinal wavelet bases also simplifies the initialization step of the discrete wavelet transform, that is, the estimation of the the fine scale scaling coefficients from the samples of a function—the estimation of $\int f(t)\phi(t-n)\,dt$ from $f(n)$. (See [33] or [31, p. 232] for overviews of initialization methods.) However, with cardinal (or interpolating) scaling functions, no such initialization step is needed. The samples $f(n)$ are themselves the values sought.

D. Shift Variance

It must be noted that if a signal $f(t)$ lies in a scaling space $V(\phi)$ or $V(\phi_0, \phi_1)$, then generally, there are translations $f(t-T)$ of the function that do not lie in the scaling space. Hence, in the multiresolution context, there is a loss of shift-invariance, which occurs in both the wavelet and the multiwavelet cases. The requirement that a function and all its shifts lie in the same scaling space is very restrictive for sampling theorems, as discussed in [14].

VII. Conclusion

The sampling issue has long been a concern in wavelets both in theory and in practice. Obtaining wavelet coefficients from a sampled signal has previously required approximation or prefiltering. However, with the new cardinal multiwavelet basis, interpolation and sampling issues are addressed without departing from orthogonal FIR multirate systems. The coefficients and the associated files for reproducing these results are available from the author or via the Internet at http://taco.poly.edu/selesi/.

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