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Formulas for Orthogonal IIR Wavelet Filters

Ivan W. Selesnick

***Abstract*—Explicit solutions are given for the rational function $P(z)$ for two classes of IIR orthogonal two-band wavelet bases, for which the scaling filter is maximally flat. $P(z)$ denotes the rational transfer function $H(z)H(1/z)$, where $H(z)$ is the (lowpass) scaling filter. The first is the class of solutions that are intermediate between the Daubechies and the Butterworth wavelets. It is found that the Daubechies, the Butterworth, and the intermediate solutions are unified by a single formula. The second is the class of scaling filters realizable as a parallel sum of two allpass filters (a particular case of which yields the class of symmetric IIR orthogonal wavelet bases). For this class, a closed-form solution is provided by the solution to an older problem in group delay approximation by digital allpole filters.**

I. Introduction

The results of this paper supplement the paper by Herley and Vetterli [8] in which orthogonal IIR wavelets are examined. (See also [29, pp. 139–141, 276–278].) This correspondence describes formulas for $P(z)$ for the design of orthogonal two-band IIR wavelets, with and without symmetry, for which the scaling filter is maximally flat. $P(z)$ denotes the rational transfer function $H(z)H(1/z)$, where $H(z)$ is the (lowpass) scaling filter. Necessarily, the associated orthogonal IIR filter banks are noncausal, which makes them substantially more difficult to use for real-time applications. However, some of the applications for which wavelet analysis has been successful—compression and denoising being two notable examples—are often performed in "batch mode" on finite length data. In such cases, noncausality is less problematic. Several authors have discussed and/or advocated the use of IIR filter banks for certain applications [1], [3], [10], [12], [15], [23], [24].

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II. Preliminaries

Let $H(z)$ denote the transfer function of the real-valued scaling filter of an orthogonal two-band wavelet basis. As such, $H(z)$ is also the lowpass filter of a two-channel orthogonal filter bank [5]. The central equation that $H(z)$ must satisfy is

$$H(z)H(1/z) + H(-z)H(-1/z) = 1. \quad (1)$$

When $H(z)$ is a polynomial solution to (1), the corresponding wavelet and scaling functions are compactly supported, and the filter bank is FIR. On the other hand, rational solutions to (1) produce infinitely supported wavelets and describe IIR filter banks that are necessarily noncausal.

It is convenient to define the function $P(z)$

$$P(z) = H(z)H(1/z). \quad (2)$$

The solutions $P(z)$ of

$$P(z) + P(-z) = 1 \quad (3)$$

are sought, where $P(z)$ permits a factorization as in (2).

This correspondence considers maximally flat solutions $H(z)$. Those are solutions for which $H(z)$ possesses the maximum number of zeros at $z = -1$ possible, given the degrees of the numerator and denominator of $H(z)$. Requiring $H(z)$ to be maximally flat is an attractive strategy for several reasons , which have been described elsewhere [5]. It is only noted here that requiring maximal flatness makes possible a straightforward design technique and produces useful wavelet bases. From this requirement, it follows that $P(z)$ has the form

$$P(z) = \frac{(1+z^{-1})^N(1+z)^N F_N(z)}{F_D(z)} \quad (4)$$

where $F_N(z)$ and $F_D(z)$ are "numerator" and "denominator" polynomials, as noted in [8]. Two special cases are well known. Certainly, the most important solution, which is described by Daubechies, has $F_D(z) = 1$. Another is the Butterworth solution [2], [6], [23]. [27] with $F_N(z) = 1$. The intermediate solutions described by Herley and Vetterli have both $F_N(z)$ and $F_D(z)$ of positive degree. However, a formula for $P(z)$ for these intermediate solutions has not been provided.

Both the Daubechies and the Butterworth solutions $P(z)$ are "halfband" instances of digital filters, which were known before the elegant theory of wavelet analysis was developed. In Section III, we introduce a formula for the intermediate solutions $P(z)$ by drawing from previous work on generalized digital Butterworth filters. The formula provided in Section III specializes to both the Daubechies and the Butterworth solutions.

In Section IV, we consider another class of orthogonal scaling filters: those realizable as a parallel sum of allpass filters (a special case of which yields symmetric wavelets). Herley and Vetterli described IIR orthogonal two-channel filter banks with symmetric filters and give examples of maximally flat solutions $H(z)$, where $H(z)$ is composed of allpass filters. This paper supplements the description given in [8] by noting that an explicit formula for $H(z)$ in this case comes directly from the solution to an older problem in group delay approximation for digital allpole filters.

III. Intermediate Daubechies-Butterworth Wavelets

In [8], Herley and Vetterli describe a class of rational solutions to (1) that are intermediate—between the compactly supported orthogonal wavelets of Daubechies [5] and the orthogonal IIR Butterworth wavelets, in the sense that each of $F_N(x)$ and $F_D(x)$ are of positive degree. These intermediate solutions can be obtained by solving a linear system of equations, as described in [8]. It turns out, however, that the generalized digital Butterworth filter provides a formula for $P(z)$.

The version of the generalized Butterworth filter used in this paper is characterized by three parameters $L, M,$ and N.

- N number of zeros of $H(z)$ at $z = -1$;
- L number of zeros that shape the passband;
- M number of poles (away from the origin).

For example, in Fig. 1, $L = 3, M = 2, N = 6$. The cut-off frequency and the roll-off rate are controlled by the triplet (L, M, N). Generalized Butterworth filters are appealing because sometimes, a filter having differing numerator and denominator degrees can achieve an improved tradeoff between performance and implementation complexity [11], [19].

Like Daubechies [5] and Herrmann [9], the generalization of the digital Butterworth filter [21] uses the transformation $x = \frac{1}{2}(1 - \cos\omega)$ with $z = e^{j\omega}$. With this change of variables, the sought rational function $P(x)$ has a lowpass behavior over the interval [0, 1]. The maximally flat behavior requires that $P(x)$ has a zero of order N at $x = 1$ (this corresponds to the stopband $\omega = \pi$). The flat behavior at $x = 0$ requires that $P(x) - 1$ has a high order zero at $x = 0$. The maximal order of that zero is $L + M + 1$. From [21], the function $P(x)$ uniquely defined by these conditions is given by

$$P(x) = \frac{(1-x)^N S(x)}{\mathcal{T}_M\{(1-x)^N S(x)\}} \tag{5}$$

where

$$S(x) = \sum_{k=0}^{L} \binom{L+M-k}{M} \binom{N-M+k-1}{k} x^k \tag{6}$$

and $\mathcal{T}_M$ denotes polynomial truncation (discarding all terms beyond the Mth power). It is assumed that M is even, that $M \geq 0, L \geq 0$, and that $N > M$. For negative values in the binomial coefficient, the convention $\binom{n+k-1}{k} = (-1)^k \binom{-n}{k}$ for $k \geq 0$ is used [18]. The first $M + 1$ coefficients of the numerator of $P(x) - 1$ are zero because those coefficients are common to both the numerator and denominator of $P(x)$. Further coefficients of the numerator of $P(x) - 1$ are simply the coefficients of $(1 - x)^N S(x)$; L of which are zero by design. Fig. 1 illustrates the maximally flat scaling filter $H(z)$ obtained using (5) with $L = 3, M = 2, N = 6$.

It should be noted that $P(x)$ is not halfband [it does not satisfy (3)] unless $N = L + M + 1$. Only then is the resulting generalized Butterworth filter a valid scaling filter.

To obtain $H(z)$ from $P(z)$, it must be possible to spectrally factor $P(z)$, as in (2). This is indeed possible because $P(x)$ is positive over the interval (0, 1). However, the transfer function $H(z)$ is most readily obtained by mapping the poles and zeros of $P(x)$ via $z = 1 - 2x \pm \sqrt{(1-2x) - 1}$. That is the standard mapping that is used, for example, in [9].

Formula (5) specializes to the well-known formula for maximally flat symmetric FIR filters that was first given in [9]. For $M = 0, N = L+1$, the halfband instance of that filter (the Daubechies polynomial) is retrieved. For $L = 0, N = M+1$, a classical halfband Butterworth filter is retrieved.

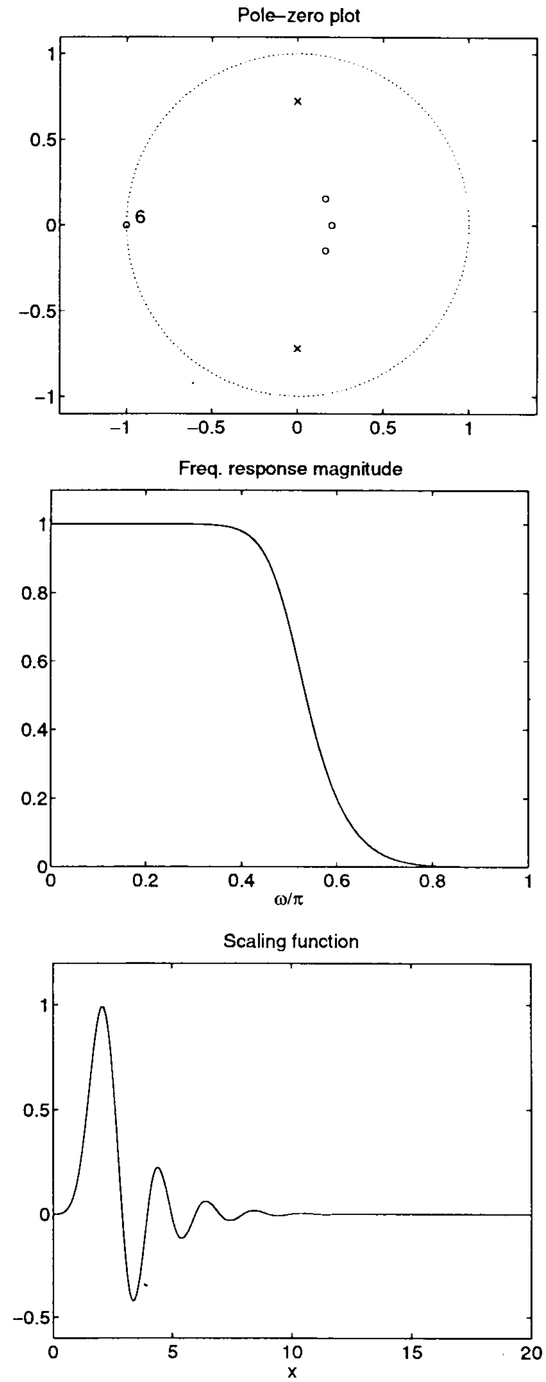

Fig. 1. Intermediate Daubechies–Butterworth scaling filter. $L = 3, M = 2, N = 6$.

Some unanswered questions remain. For the polynomial (FIR) solution, the following has been shown [22].

i) The zeros of $H(z)$, other than those at $z = -1$, lie near $|1 + z| = \sqrt{2}$.
ii) The slope of $|H(\omega)|^2$ is about $\sqrt{N/\pi}$ at $\omega = \pi/2$.
iii) The transition from 1 to 0 has a width of about $2\sqrt{2/N}$.

What are the analogous results for the rational (IIR) solution?

IV. Wavelet Filters from Allpass Sums

It is widely appreciated that the only polynomial solution to (1) that produces a real-valued orthogonal basis of *symmetric* wavelets, is the

Haar solution. To obtain a two-band wavelet basis of symmetric real-valued functions with more regularity than the Haar solution without giving up orthogonality requires the basis functions and filters have infinite support. To this end, Herley and Vetterli gave the form for $H(z)$ as

$$H(z) = \tfrac{1}{2}\,[A(z^2) + z^{-1}A(z^{-2})] \tag{7}$$

where $A(z)$ is an allpass filter of degree N, with N even. This is a parallel sum of allpass filters, which is a structure that has been well studied. Such filters can be implemented with low complexity structures that are robust to finite precision effects [17]. Noting that $A(z)/A(z^{-1}) = A^2(z)$, it is straightforward to verify that $H(z)$ in (7) satisfies (1). Note that $H(z)$ has half-sample symmetry[1]. Because the stable noncausal impulse response of $H(z)$ is symmetric, $H(z)$ produces symmetric wavelets. An example is provided in [8]. Some properties of symmetric wavelet bases are also discussed in [4].

Given the form (7), an allpass filter $A(z)$ is sought so that $H(z)$ is maximally flat. An explicit solution exists and is supplied by the solution to a group delay approximation problem. To introduce this connection, note that the magnitude response $|H(e^{j\omega})|$ of (7) is not affected by multiplication of $H(z)$ by an allpass function. In particular, $H(z)$ can be divided by the allpass function $A(z^{-2})$ without affecting the number of zeros of $H(z)$ at $z = -1$. Therefore, we can consider, instead of (7), the sum

$$\tfrac{1}{2}\,[A^2(z^2) + z^{-1}]. \tag{8}$$

It is clear that if $A^2(z^2) \approx -z^{-1}$ at $z = -1$, then the desired lowpass behavior of $H(z)$ is obtained. It follows that the group delay of the allpass function $A^2(z^2)$ should be 1 at $\omega = \pi$. In turn, it follows that the group delay of $A(z^2)$ should be one half at $\omega = \pi$; therefore, the group delay of $A(z)$ should be one fourth at $\omega = 0$. Writing the allpass function $A(z)$ as

$$A(z) = \frac{z^{-N}D(1/z)}{D(z)} \tag{9}$$

where $D(z)$ is of degree N, we finally find that the group delay of the digital allpole filter $1/D(z)$ should approximate[2] $\tau = 1/8 - N/2$ at $\omega = 0$. To obtain the maximally flat $H(z)$, given the form (7), the maximum number of derivatives of the group delay of $1/D(z)$ should be made to vanish at $\omega = 0$. The solution is given by the digital allpole filter [7], [26], the group delay of which is maximally flat at $\omega = 0$

$$\frac{1}{D(z)} = \frac{1}{\displaystyle\sum_{n=0}^{N} a_n z^{-n}} \tag{10}$$

where

$$a_n = (-1)^n \binom{N}{n} \frac{(2\tau)_n}{(2\tau + N + 1)_n}. \tag{11}$$

The value of the group delay at $\omega = 0$ is τ. The pochhammer symbol $(x)_n$ denotes the rising factorial $(x)_n = (x)\cdot(x+1)\cdot(x+2)\cdot s(x+n-1)$. Interestingly, (11) is also useful for the construction of *biorthogonal* wavelet bases, where both the analysis and synthesis IIR filters are stable causal [14].

The solution (11) is generalized in [20]. The group delay of the solution (11) has a flat characteristic at $z = 1$ only. Achieving a flat delay characteristic at both $z = 1$ and $z = -1$, where the degrees of flatness at $z = 1$ and $z = -1$ are not necessarily the same, is considered in [20]. An explicit solution to that problem is given and is used for the design of lowpass filters (not necessarily halfband) realizable as a parallel sum of two allpass filters.

[1] Half-sample symmetry is symmetry of the form $h(n) = h(n_o - n)$, where n_o is an odd integer.

[2] Note that if the group delay of $1/D(z)$ is τ, then the group delay of $A(z)$ is $2\tau + N$. Therefore, if the group delay of $A(z)$ is to be X, then the group delay of $1/D(z)$ is to be $(X - N)/2$.

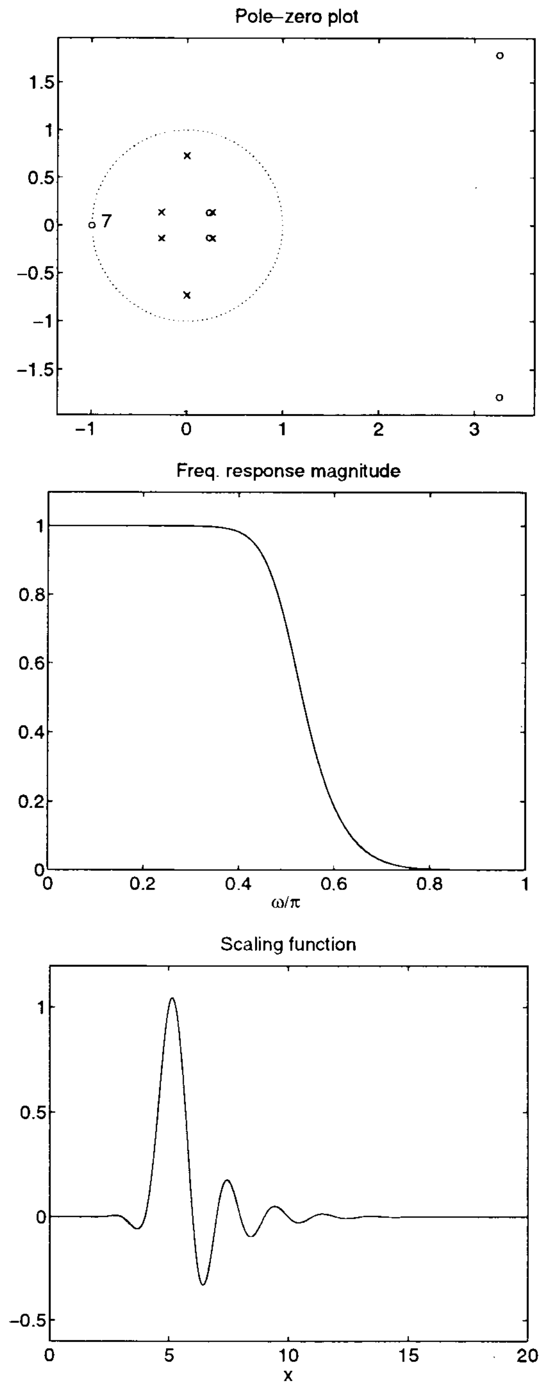

Fig. 2. Maximally flat scaling filter realizable as the sum of an allpass filter and a pure delay. $H(z) = \frac{1}{2}[A(z^2) + z^{-5}]$, where the degree of $A(z)$ is 3.

A. More General Allpass Sums

It is straightforward to show that, in fact, any sum

$$H(z) = \tfrac{1}{2}\,[A_1(z^2) + z^{-1}A_2(z^2)] \tag{12}$$

where $A_1(z)$ and $A_2(z)$ are allpass filters, satisfies (1). Again, the maximally flat delay filter $1/D(z)$ described above (with appropriate DC group delay τ) provides the solution. For example, in [30], the

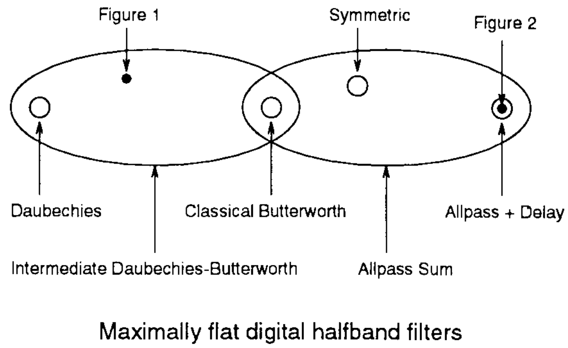

Fig. 3. Venn diagram illustrating relationships among maximally flat orthogonal IIR scaling filters (halfband filters).

linear system of equations that must be solved to obtain the maximally flat solution, when $A_2(z)$ is a pure delay z^{-d}, is described, but no explicit solution was given. In this case, when $H(z) = (A(z^2) + z^{-(2d+1)})/2$, the allpass filter $A(z)$ is given by (9)–(11), but now with $\tau = d/2 + 1/4 - N/2$. For $N = 3, d = 2$, Fig. 2 illustrates the scaling filter $H(z)$ and the scaling function. Interestingly, frequency selective filters realizable as a parallel sum of an allpass filter and a pure delay have approximately linear phase in the passband.

It should also be noted that the classical Butterworth filter can be realized as a parallel sum of allpass filters. Indeed, the classical lowpass (Butterworth, Chebyshev I and II, and elliptic) transfer functions can each be realized as such [28]. In fact, orthogonal IIR filter banks with elliptic filters, realized as allpass sums, have been described in [13]. It should be emphasized that the allpass sum scaling filters described in this paper are maximally flat *with respect to* their structure. The Daubechies and the intermediate Daubechies–Butterworth filters of Section III are not realizable as allpass sums.

V. Conclusion

The maximally flat FIR, IIR, and "allpass sum" orthogonal scaling filters are obtained as instances of maximally flat digital filters. While this was well known for the Daubechies FIR and Butterworth IIR solutions, the recognition that it is also true for

i) the intermediate Daubechies-Butterworth of [8];
ii) the symmetric IIR solutions of [8];
iii) the approximately linear-phase IIR solution of [30];
iv) allpass sums of the form (12)

makes available explicit formulas for $P(z)$. In case i), the generalized digital Butterworth filter of [21] provides the solution. In cases ii)–iv), the digital allpole filter [7], [26] for which the group delay is maximally flat at DC, provides the solution. Fig. 3 clarifies the relationships among these halfband filters. Further work includes extending these results to the M-band case, as was done in [25] for FIR scaling filters.

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