Balanced GHM-Like Multiscaling Functions

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***Abstract*— The Geronimo–Hardin–Massopust (GHM) multiwavelet basis exhibits symmetry, orthogonality, short support, and approximation order $K=2$, which is not possible for wavelet bases based on a single scaling-wavelet function pair. However, the filterbank associated with this basis does not inherit the zero moment properties of the basis. This work describes a version of the GHM multiscaling functions (constructed with Gröbner bases) for which the zero moment properties do carry over to the associated filterbank. That is, the basis is balanced up to its approximation order $K=2$.**

***Index Terms*— Filterbank, FIR, Gröbner bases, orthogonal multiwavelet.**

I. Introduction

IT IS OFTEN desired, in image processing for example, that the scaling function of a wavelet basis be symmetric and of compact support. However, as is well known, except for the Haar basis, the scaling function $\phi(t)$ of an orthogonal real-valued wavelet basis can not have both symmetry and compact support. For this reason, the orthogonal multiwavelet basis (a wavelet basis based on more than one scaling function) constructed by Geronimo *et al.* (GHM) in [3] is most interesting. It is based on two symmetric scaling functions $\phi_0(t)$, $\phi_1(t)$, both of compact support. A symmetric/antisymmetric pair of wavelets $\psi_0(t)$, $\psi_1(t)$ were developed in [2] and [9]. The basis also has approximation order two [2 zero wavelet moments: $\int t^k \psi_i(t)\,dt = 0$, $k=0, 1$, $i=0, 1$].

Unfortunately, the filterbank associated with a multiwavelet basis does not inherit the zero moment properties of the basis, unless the basis satisfies additional properties [6], [7]. In other words, for multiwavelet bases, zero moments of the continuous-time wavelets $\psi_i(t)$, defined on $\mathbb{R}$, do not imply zero moments of the discrete-time wavelet filters $h_i(n)$, defined on $\mathbb{Z}$. Multiwavelet bases for which the zero moment properties *do* carry over to the discrete-time filterbank are called *balanced* after Lebrun and Vetterli [5]. Specifically, multiwavelet bases for which the associated filter bank preserves/annihilates the set $\mathcal{P}_{K-1}$ of polynomials of degree $k<K$ are said to be *order-K balanced.*

The GHM basis has approximation order two, but unfortunately is order zero balanced. The filterbank associated with the GHM basis does not preserve/annihilate constant signals. This letter describes orthogonal GHM-like multiscaling functions balanced up to their approximation order $K=2$. Like the GHM basis:

1) both $\phi_0(t)$ and $\phi_1(t)$ are symmetric;
2) both $h_0(n)$ and $h_1(n)$ are symmetric, odd length (type 1) finite impulse respoonse (FIR) filters;
3) $h_0(n)$, $h_1(n)$ differ in length by four.

II. Preliminaries

This paper considers multiwavelet bases based on two scaling functions $\phi_0(t)$, $\phi_1(t)$ and two wavelet functions $\psi_0(t)$, $\psi_1(t)$. Accordingly, there are two scaling filters $h_0(n)$, $h_1(n)$ and two wavelet filters $h_2(n)$, $h_3(n)$.

The functions $\phi_0(t)$, $\phi_1(t)$ are *orthogonal multiscaling functions* if

1) $\phi_0(t)$, $\phi_1(t)$ satisfy a *matrix* dilation equation

$$\underline{\phi}(t) = \sqrt{2}\sum_n C(n)\,\underline{\phi}(2t-n)$$

where $\underline{\phi}(t) = (\phi_0(t), \phi_1(t))^t$, and $C(n)$ are 2×2 matrices.

2) $\phi_0(t)$, $\phi_1(t)$ are orthogonal to their integer shifts:

$$\int \phi_i(t)\phi_j(t-n)\,dt = \delta(i-j)\cdot\delta(n).$$

The notation for $C(n)$ used in this paper is $[C(n)]_{i,j} = h_i(2n+j)$. For example

$$C(0) = \begin{pmatrix} h_0(0) & h_0(1) \\ h_1(0) & h_1(1) \end{pmatrix}, \quad C(1) = \begin{pmatrix} h_0(2) & h_0(3) \\ h_1(2) & h_1(3) \end{pmatrix}$$

etc., where h_0 and h_1 are the two scaling filters.

For h_0, h_1 to generate orthogonal multiscaling functions ϕ_0, ϕ_1, it is necessary that they must be orthogonal to their shifts by four:

$$\sum_n h_i(n)\,h_j(n+4k) = \delta(i-j)\cdot\delta(k). \tag{1}$$

Order-one balanced multifilterbanks preserve/annihilate constant signals. From [7] the condition for order one balancing is

$$(z^{-3}+z^{-2}+z^{-1}+1) \quad \text{divides} \quad H_0(z)+H_1(z). \tag{2}$$

Order-two balanced multifilterbanks preserve and annihilate ramp and constant signals. The condition for order-two balancing is

$$(z^{-3}+z^{-2}+z^{-1}+1)^2 \quad \text{divides} \quad H_0(z)+\left(\frac{3-z^{-4}}{2}\right)H_1(z). \tag{3}$$

Conditions for order-K balancing are given in [6] and [7].

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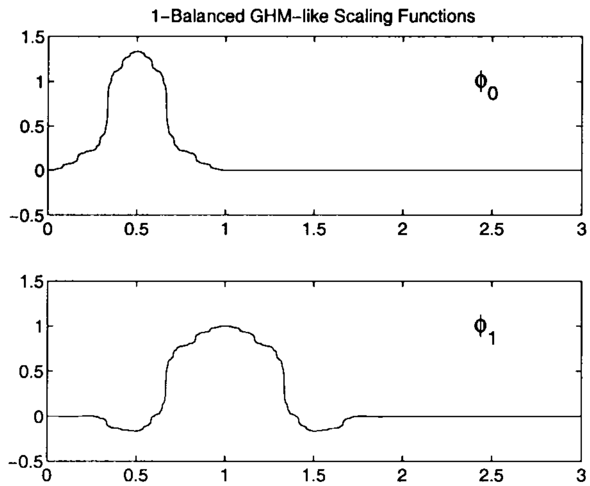

Fig. 1. Symmetric order one balanced orthogonal multiscaling functions, supported on [0, 1] and [0, 2].

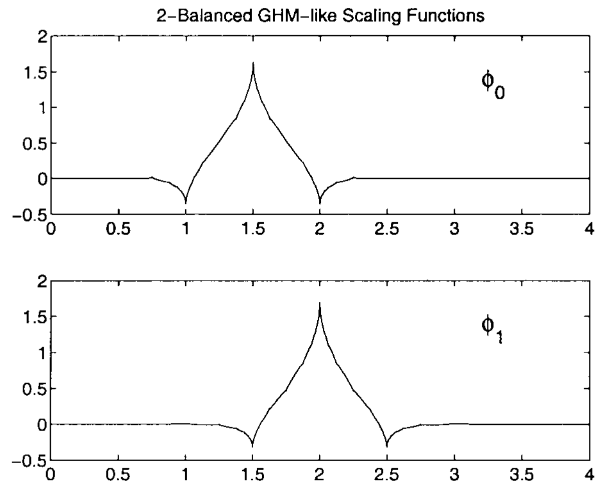

Fig. 2. Symmetric order-two balanced orthogonal multiscaling functions, supported on [0, 3] and [0, 4].

III. Balanced GHM-Like Multiscaling Functions

Our problem is to find symmetric h_0 and h_1 such that they satisfy the orthogonality conditions (1) and the balancing conditions (3). This is a system of nonlinear equations—the balancing conditions (3) are linear, but the orthogonality conditions (1) are quadratic. We use a lexical Gröbner basis to solve the nonlinear equations. Given a system of multivariate polynomials, a Gröbner basis (GB) is a new set of multivariate polynomial equations, having the same set of solutions [1]. When the lexical ordering of monomials is used, and there are a finite number of solutions, the "last" equation of the GB will be a polynomial in a single variable—so its roots can be computed. These roots can be substituted into the remaining equations, etc.—like back substitution in Gaussian elimination for linear equations. Unfortunately, GB solutions are only practical for small problems, because computing a GB is highly compute and memory intensive. However, for certain problems, Gröbner bases are very useful. See, for example, [8] for a description of a previous application of Gröbner bases to filter design.

The GHM scaling filters $h_0(n), h_1(n)$ are of lengths 3 and 7. Because balancing is a stronger condition than is approximation order (balance order-K implies approximation order-K), it cannot be expected that order-two balanced GHM-like scaling filters are possible for these lengths. For the lengths (3, 7) it is possible to increase the balance order by one, only by decreasing the approximation order by one. The result is an orthogonal system with order-one approximation and order-one balancing, shown in Fig. 1.

By increasing the lengths to (7, 11), it is possible to obtain a GHM-like multiscaling functions having approximation order-two and order-two balancing, as illustrated in Fig. 2. For the scaling filter coefficients, the algebraic solution in terms of radicals shown in Table I was obtained using Gröbner bases (for the computation of which, the software *singular* was employed [4]).

TABLE I
Order-Two Balanced Orthogonal Symmetric Multiscaling Filters. $h_i(n)$ are Normalized to Sum to One

$A = 8/17 + 3\sqrt{2}/34$
$B = -A/2 + 1/2 + \sqrt{10404A^2 - 10812A + 2907}/204$
$h_0(0) = h_0(6) = B + 2A/3 - 2/3$
$h_0(1) = h_0(5) = -2B - 7A/6 + 7/6$
$h_0(2) = h_0(4) = B$
$h_0(3) = A$
$h_1(0) = h_1(10) = 17AB/24 - 11B/24 - A/3 + 5/24$
$h_1(1) = h_1(9) = -17AB/12 + 11B/12 + 7A/12 - 35/96$
$h_1(2) = h_1(8) = 17AB/24 - 11B/24$
$h_1(3) = h_1(7) = -A/2 + 1/4$
$h_1(4) = h_1(6) = 23/24 - 17AB/12 - 13B/12 - A/3$
$h_1(5) = -53/48 + 17AB/6 + 13B/6 + 7A/6$

IV. Conclusion

This letter presented a balanced version of the GHM basis—the scaling functions are simultaneously symmetric, orthogonal and of compact support. In addition, the associated filterbank also has the zero moments properties, important for compression and denoising. Programs for reproducing these results will be available at http://taco.poly.edu/selesi.

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