Low-Pass Filters Realizable as All-Pass Sums: Design via a New Flat Delay Filter

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***Abstract*—This paper describes a new class of maximally flat low-pass recursive digital filters. The filters are realizable as a parallel sum of two all-pass filters, a structure for which low-complexity low-noise implementations exist. Note that, with the classical Butterworth filter of degree N which is retrieved as a special case, it is not possible to adjust the delay (or phase linearity). However, with the more general class of filters described in this paper, the adjustment of the delay becomes possible, and the tradeoff between the delay and the phase linearity can be chosen. The construction of these low-pass filters depends upon a new maximally flat delay allpole filter, for which the degrees of flatness at $\omega = 0$ and $\omega = \pi$ are not necessarily equal. For the coefficients of this flat delay filter, an explicit solution is introduced, which also specializes to a previously known result.**

I. Introduction

The design of digital all-pass filters has received much attention in recent years, for it has become well known that: 1) low-complexity structures with low roundoff noise behavior are available for all-pass filters [32], [43] and 2) they are useful components in a variety of applications. Indeed, while traditional applications of all-pass filters appear to be phase equalization [7], [19] and fractional delay elements [18], their uses in multirate filtering, filterbanks, notch filtering, recursive phase splitters, and Hilbert transformers have also been described [30], [35], [37], [38]. Of particular interest has been the design of frequency selective filters realizable as a parallel combination of two all passes

$$H(z) = \frac{1}{2}[A_1(z) + A_2(z)], \tag{1}$$

illustrated in Fig. 1. This structure, the all-pass sum, has its history in analog lattice circuitry [45] and wave-digital filters [9].

It is interesting to note that digital filters of odd degree, obtained from the classical analog (Butterworth, Chebyshev, and elliptic) prototypes via the bilinear transformation, can be realized as all-pass sums[1] [34], [42]–[44]. As all-pass sums, such filters can be realized with low complexity structures that are robust to finite precision effects [43]. However, the realization of a low-pass filter as an all-pass sum is not restricted to the classical filter prototypes. The all-pass sum is a useful generalization of the classical transfer functions, that is honored with a number of benefits. Specifically, when some degree of phase linearity is desired, nonclassical filters of the form (1) can be designed that achieve superior results with respect to implementation complexity and phase linearity [11], [14].

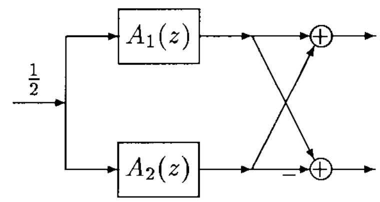

Fig. 1. All-pass sum. The upper branch gives $1/2[A_1(z) + A_2(z)]$. The lower branch gives the complementary filter $1/2[A_1(z) - A_2(z)]$ without additional filtering.

The desired degree of phase linearity can, in fact, be structurally incorporated. If one of the all-pass branches in an all-pass sum is a pure delay z^n, then the all-pass sum exhibits approximately linear phase in the passbands [17], [31]. The frequency selectivity is then obtained by appropriately designing the remaining all-pass branch. Interestingly, by varying the number of delay elements used and the degrees of $A_1(z)$ and $A_2(z)$, the phase linearity can be affected. Simultaneous approximation of the phase and magnitude is a difficult problem in general, so the ability to structurally incorporate this aspect of the approximation problem is attractive.

While general procedures for all-pass design [4], [10], [15], [16], [21], [24], [26]–[28], [36], [46] are applicable to the design of frequency selective all-pass sums, several publications have addressed, in addition to the general problem, the details specific to all-pass sums [1], [2], [13], [22], [31], [33], [35]. Several authors have also described iterative Remez-like exchange algorithms for the design of all-pass filters and all-pass sums according to the Chebyshev criterion [11], [14], [20], [38].

This paper considers the design of maximally flat low-pass filters realizable as all-pass sums. It is explained that the design problem can be formulated as the problem of group delay approximation of a single allpole filter[2]. The solution sought in this paper depends on an allpole filter, the group delay of which possesses a maximally flat characteristic at $\omega = 0$ and $\omega = \pi$. An explicit solution is given for this

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[1] The classical filters of even degree can also be realized as a sum of two all-pass filters, however, in that case complex coefficients are required. In this paper, only real-coefficient filters will be considered.

[2] In this paper, the term "allpole filter" allows for zeros at $z = 0$.

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allpole filter, where the degrees of flatness at $\omega = 0$ and $\omega = \pi$ need not be equal. Also, by developing a method by which the cutoff frequency can be continuously varied, it is found that the classical Butterworth digital filter is retrieved a special case of the class of low-pass filters described in this paper. Various examples are provided and simple Matlab programs are given in the Appendix. The solution given is applicable to the approximately linear phase case and the general case.

II. Flat Delay Filter

Let $G(\omega)$ denote the group delay, the negative derivative of the phase with respect to ω. This section examines the transfer function

$$\frac{b_0}{\sum_{n=0}^{N} a_n z^{-n}} = \frac{b_0}{D(z)} \tag{2}$$

where $a_0 = 1$ and the dc gain is normalized to unity: $b_0 = \sum_{n=0}^{N} a_n$. The goal is to design the transfer function so that its group delay possesses a specified degree of flatness at $\omega = 0$ and $\omega = \pi$, where the degree of flatness at each point is not necessarily equal. There are N free parameters $a_1, \cdots, a_N$. K will denote the number of constraints assigned to $\omega = 0$, and L will denote the number of constraints assigned to $\omega = \pi$. The problem is formulated as follows.

Given τ (the desired group delay), K, L, and N, with $K + L = N$, find N real coefficients a_n such that the group delay of (2) satisfies the following derivative constraints:

1) $G(\omega = 0) = \tau$. (If $K > 0$);
2) $G(\omega = \pi) = \tau$. (If $L > 0$);
3) $G^{(2i)}(\omega = 0) = 0$, for $i = 1, \cdots, K - 1$;
4) $G^{(2i)}(\omega = \pi) = 0$, for $i = 1, \cdots, L - 1$.

Because the group delay is an even function of ω, the odd indexed derivatives of $G(\omega)$ are automatically zero, so they need not be specified. The solution has the property that $G^{(i)}(\omega = 0) = 0$ for $i = 1, \cdots, 2K - 1$, and $G^{(i)}(\omega = \pi) = 0$ for $i = 1, \cdots, 2L - 1$.

Following the derivation given in [41], the four group delay conditions are equivalent to the following system of equations, linear in the coefficients a_n:

$$\sum_{n=1}^{N} (n + \tau)^{2k+1} a_n = -\tau^{2k+1}, \quad \text{for } k = 0, 1, \cdots, K - 1 \tag{3}$$

$$\sum_{n=1}^{N} (-1)^n (n + \tau)^{2l+1} a_n = -\tau^{2l+1}, \quad \text{for } l = 0, 1, \cdots, L - 1. \tag{4}$$

The system matrix is comprised of two Vandermonde-like matrices, corresponding to (3) and (4), respectively. Vandermonde matrices arise in polynomial interpolation problems and, accordingly, become ill-conditioned in cases where many derivatives at a few points are specified, as is the case here. With some assistance from the computer algebra system [6], the solution was found to be

TABLE I
Maximally Flat Delay Digital Allpole Filter, $K = 6$, $L = 3$, $\tau = 7/2$ and $\tau = -3/2$

n	a_n	$a_n, (\tau = 7/2)$	$a_n, (\tau = -3/2)$
0	1	1	1
1	$-\frac{3\tau}{5+\tau}$	$-\frac{21}{17}$	$\frac{9}{7}$
2	$-\frac{36\tau}{(2\tau+11)(5+\tau)}$	$-\frac{14}{17}$	$\frac{27}{14}$
3	$\frac{(16\tau+116)(1+\tau)\tau}{(2\tau+11)(5+\tau)(6+\tau)}$	$\frac{602}{323}$	$\frac{23}{42}$
4	$-\frac{(6+6\tau)(2\tau-3)\tau}{(2\tau+11)(5+\tau)(6+\tau)}$	$-\frac{84}{323}$	$\frac{3}{14}$
5	$-\frac{6\tau(2\tau+21)(2+\tau)(1+\tau)}{(5+\tau)(2\tau+11)(7+\tau)(6+\tau)}$	$-\frac{308}{323}$	$-\frac{9}{154}$
6	$\frac{(16\tau+28)(2+\tau)(1+\tau)\tau}{(5+\tau)(2\tau+11)(7+\tau)(6+\tau)}$	$\frac{154}{323}$	$\frac{1}{462}$
7	$\frac{(108+36\tau)(2+\tau)(1+\tau)\tau}{(2\tau+11)(8+\tau)(7+\tau)(6+\tau)(5+\tau)}$	$\frac{858}{7429}$	$\frac{9}{2002}$
8	$-\frac{3\tau(2\tau+7)(3+\tau)(2+\tau)(1+\tau)}{(2\tau+11)(8+\tau)(7+\tau)(6+\tau)(5+\tau)}$	$-\frac{1001}{7429}$	$-\frac{3}{2002}$
9	$\frac{\tau(2\tau+7)(4+\tau)(3+\tau)(2+\tau)(1+\tau)}{(2\tau+11)(9+\tau)(8+\tau)(7+\tau)(6+\tau)(5+\tau)}$	$\frac{1001}{37145}$	$\frac{1}{6006}$

$$a_n = \frac{(-1)^n}{(2\tau + K + L + 1)_n} \binom{K+L}{n} \sum_{i=0}^{L} (-4)^i \times \binom{L}{i} \frac{(\tau)_i (n - i + 1)_i (2\tau + 2i)_{n-i}}{(K + L + 1 - i)_i} \tag{5}$$

for $0 \leq n \leq N$. The Pochhammer symbol $(x)_n$ denotes the rising factorial $(x)_n = (x) \cdot (x + 1) \cdot (x + 2) \cdots (x + n - 1)$. We define $(x)_n$ as 1 for $n \leq 0$. Although (5) looks rather cumbersome, the coefficients a_n can be efficiently calculated. A method for doing so is described in the appendix, where a Matlab program for this calculation is also given.

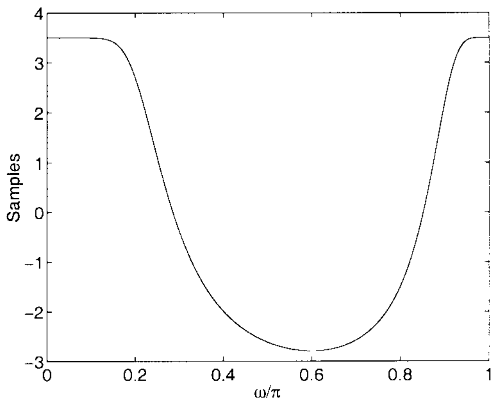
Note that for some negative values of τ no solution exists. These are values for which the denominator in (5) vanishes: $\tau = -(1/2)(K + L + n)$ for $n = 1, \cdots, N$. These values of τ are more negative than will be needed later on. It is expected that there should be N values of τ for which no solution exists, since the determinant of the system matrix is a polynomial in τ of degree N. Except for those N values, τ can be any real number, positive or negative, although $1/D(z)$ will be unstable for sufficiently negative values of τ.

The use of (5) for negative values of τ is not purely for theoretical interest. In fact, for the design of stable causal low-pass filters realizable as all-pass sums, the use of unstable solutions (5) obtained with negative values of τ will be necessary, as illustrated in Section III. Properties of the zeros of $1/D(z)$ that will later be relevant are discussed in Appendix A.

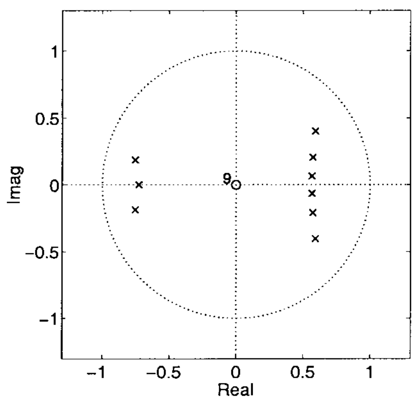
Example 1: To illustrate the maximally flat delay filter, a ninth degree example is provided. For this example, $N = 9$, $K = 6$, and $L = 3$. The coefficients a_n as rational functions of τ are given in Table I, as well as the solution for $\tau = 7/2$ and $\tau = -3/2$. The group delay and the pole–zero diagram for those two filters are shown in Figs. 2 and 3. For $\tau = 7/2$, the solution is stable. For $\tau = -3/2$, the solution possesses two poles outside the unit circle.

A. A Special Case

The allpole filter possessing a flat delay characteristic only at dc $(L = 0)$ is well established in the literature. In this case,

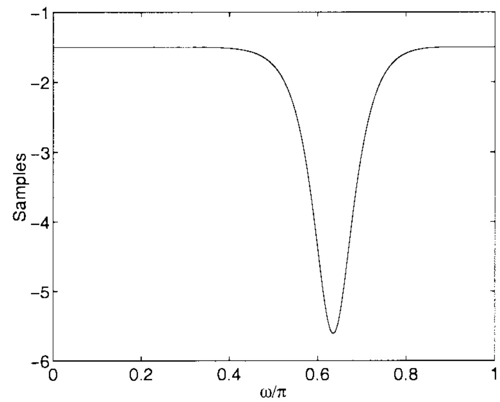


(a)

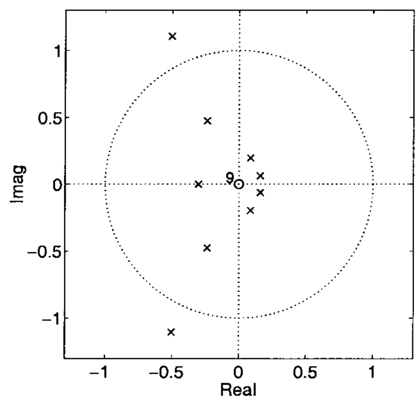


(b)

Fig. 2. Maximally flat delay filter, $K = 6,\ L = 3,\ \tau = 7/2$; group delay and pole–zero diagram.



(a)



(b)

Fig. 3. Maximally flat delay filter, $K = 6,\ L = 3,\ \tau = -3/2$; group delay and pole–zero diagram.

the solution (5) specializes to the previously reported solution

$$a_n = (-1)^n \binom{N}{n} \frac{(2\tau)_n}{(2\tau + N + 1)_n}. \tag{6}$$

This solution has been described in various forms, see for example [8], [12], [40], [41]. For example, in [41] this formula is written as

$$a_n = (-1)^n \binom{N}{n} \prod_{i=0}^{N} \frac{2\tau + i}{2\tau + n + i} \tag{7}$$

which is the same as (6) after simplification (cancelling the $N + 1 - n$ terms common to the numerator and denominator of the $N + 1$ term product). It should also be noted that the solution of [40] is the same when it is understood that [40] considers the transfer function $z^{-N}/D(z)$; the appropriate change of variables is straightforward.

Various uses of the special case (6) have been described. One application of (6) is the design of all-pass fractional delay filters [18]. It is also used in one of several approaches described in [23] for the design of negative group delay systems. It should also be noted that if one sets $K = L$ in (5), then $D(z)$ is an even function of z. In this case, one can write $D(z) = E(z^2)$, where $E(z)$ can be obtained using (6). Hence, with the substitution $z \to z^2$, the special case $L = 0$ is mapped to the special case $K = L$. That solution is also useful for the design of halfband filters [37] and has been developed and used for the design of biorthogonal filterbanks and wavelet bases in [29] (see also [3]).

It should be noted that the bilinear transformation (BLT) does not preserve the flat delay characteristic. Therefore, the Bessel filter (the analog filter with a maximally flat delay characteristic at dc) does not provide, via the BLT, the digital filter (6) having that characteristic.

III. The All-Pass Sum

It will be useful to write the all-pass sum as

$$\frac{1}{2}[z^{-d} A_2(z) + A_1(z)] \tag{8}$$

where $A_1(z)$ and $A_2(z)$ are both stable causal all passes and d is a nonnegative integer. Note that the all-pass sum

$$\frac{1}{2}\left[z^{-d} + \frac{A_1(z)}{A_2(z)}\right] \tag{9}$$

has the same magnitude response as (8), but has a different phase response and has poles outside the unit circle. In the design of all-pass sums for magnitude approximation, it is therefore sufficient, as noted in [11], to consider only the design of an all-pass sum of the form

$$\frac{1}{2}[z^{-d} + A(z)] \tag{10}$$

where $A(z)$ may possess poles outside the unit circle in general. The factorization of $A(z)$ into $A_1(z)/A_2(z)$, where $A_1(z)$ and $A_2(z)$ are both stable causal all passes, is straightforward. With such a factorization, the transfer function (8) will be stable causal. For this reason, stability of $A(z)$ is unimportant during the design phase—a causal stable low-pass filter of the form (8) can be simply obtained even when $A(z)$ is unstable. If $A(z)$ is itself stable causal, then the total response has approximately linear phase, as previously mentioned. The degree of phase linearity of the total response depends, in part, on the relative degrees of $A_1(z)$ and $A_2(z)$. For greater values of d the phase will be more linear but the overall delay will be greater. As will be illustrated by examples, this tradeoff between phase linearity and overall delay can be controlled by appropriate selection of d and design of $A(z)$.

Consider the design of an all-pass sum of the form (10) where $A(z)$ is an all-pass filter of degree N with real coefficients. Being all pass, $A(z)$ is of the form

$$A(z) = \frac{z^{-N}D(1/z)}{D(z)} \tag{11}$$

where $D(z)$ is a real polynomial in z^{-1} of degree N. In addition, if $A(z)$ satisfies the following approximations, then the total response of (10) will clearly be low pass.

1) $A(z) \approx z^{-d}$ in the passband.
2) $A(z) \approx -z^{-d}$ in the stopband.

It follows that the group delay of $A(z)$ should approximate d in both the passband and the stopband. Consequently, the group delay of the transfer function $1/D(z)$ should approximate $(d-N)/2$ in both the passband and the stopband. (Note that if the group delay of $1/D(z)$ is τ, then the group delay of $A(z) = z^{-N}D(1/z)/D(z)$ is $2\tau + N$. Therefore, if the group delay of $A(z)$ is to be X, then the group delay of $1/D(z)$ is to be $(X-N)/2$. Various useful expressions for the group delay are provided in [34].

The solution sought in this paper depends upon the transfer function $1/D(z)$, the group delay of which approximates $(d-N)/2$ in the maximally flat sense. This solution is obtained by setting derivatives of the group delay equal to zero at $\omega = 0$ and $\omega = \pi$, and was introduced in Section II. The low-pass filters described herein are maximally flat in the following sense. Given the all-pass sum structure (8) where d and N are specified, the coefficients are chosen so that as many derivatives of the magnitude response as possible vanish at $\omega = 0$ and $\omega = \pi$. As in Section II, K will denote the number of constraints assigned to $\omega = 0$, and L will denote the number of constraints assigned to $\omega = \pi$. The relative values of K and L determine the location of the transition region. As above $N = K + L$.

Some restrictions apply to the selection of d, K, and L. Leaving the details in Appendix B, d, K, and L must be selected so that they satisfy the following two conditions:

1) $|K-L| + 1 \le d \le K + L + 1$;
2) d must have the same parity as $K + L + 1$.

It follows that $\tau = (d-N)/2$ must be an odd multiple of one half. Also, when K and L are approximately equal, more choices for d are available than when K and L are disparate. Note that, for fixed values of K and L, d can take on $(K + L - |K-L|)/2 + 1$ different values. Similarly, for a fixed value of d, there are d different permissible (K, L) pairs.

As noted previously, the all pass $A(z)$ will in general possess poles lying outside the unit circle, thus requiring the factorization $A(z) = A_1(z)/A_2(z)$ where $A_1(z)$ and $A_2(z)$ are both stable causal. The degree of $A_2(z)$ is equal to the number of poles of $A(z)$ outside the unit circle, while the degree of $A_1(z)$ is N minus this number. Denote the degrees of $A_1(z)$ and $A_2(z)$ by n_1 and n_2, respectively. From the observations in Appendix A, one has

$$n_2 = 2\left\lfloor \frac{N-d+1}{4} \right\rfloor \tag{12}$$

$$n_1 = N - 2\left\lfloor \frac{N-d+1}{4} \right\rfloor. \tag{13}$$

The important special case $d = N-1$ yields a low-pass filter with approximately linear phase at $\omega = 0$; for $d = N-1$, one has $n_2 = 0$ and no factorization of $A(z)$ is necessary. In this case, $H(z) = (z^{-d} + A(z))/2$ is stable causal. Although the same is true for $d = N+1$, the corresponding low-pass filters have a significantly poorer magnitude response.

To be explicit, the stable causal $H(z)$ is obtained as follows. Let z_i, for $i = 1, \cdots, N$, denote the zeros of $D(z)$. Then $D_1(z)$ and $D_2(z)$ are, respectively, formed from the zeros of $D(z)$ that lie inside and outside the unit circle

$$D_1(z) = \prod_{|z_i|<1} \left(z^{-1} - z_i^{-1}\right) \tag{14}$$

$$D_2(z) = \prod_{|z_i|>1} \left(z^{-1} - z_i\right). \tag{15}$$

Then the all-pass subfilters are given by

$$A_1(z) = \frac{z^{-n_1}D_1(1/z)}{D_1(z)} \tag{16}$$

$$A_2(z) = \frac{z^{-n_2}D_2(1/z)}{D_2(z)}. \tag{17}$$

Define $\tilde{D}_1(z) = z^{-n_1}D_1(1/z)$ and $\tilde{D}_2(z) = z^{-n_2}D_2(1/z)$. The all-pass sum is then

$$H(z) = \frac{1}{2}\left[z^{-d}\frac{\tilde{D}_2(z)}{D_2(z)} + \frac{\tilde{D}_1(z)}{D_1(z)}\right] \tag{18}$$

$$= \frac{1}{2}\left[\frac{z^{-d}\tilde{D}_2(z)D_1(z) + \tilde{D}_1(z)D_2(z)}{D_2(z)D_1(z)}\right]. \tag{19}$$

From (19), it is evident that the numerator of $H(z)$ is symmetric and of degree $N + d$. It is worth noting that $H(z)$ can be implemented as an all-pass sum with N multiplications per sample and $N + d$ delay elements. (An all pass of degree n can be implemented with n multiplications per sample and n delay elements [25].)

Example 2: To illustrate the design of a low-pass filter realizable as an all-pass sum, consider an example where

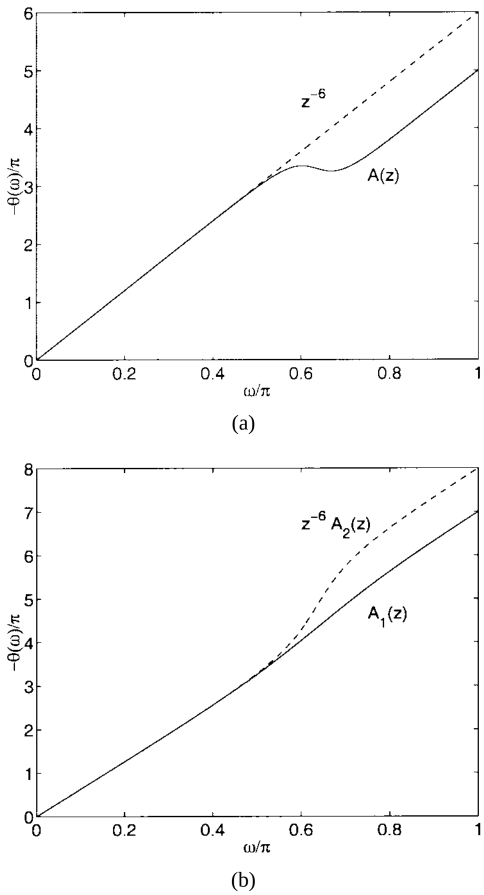


Fig. 4. The unwrapped phase responses of z^{-6} and $A(z)$ gives rise to the desired low-pass behavior of the sum $(z^{-6} + A(z))/2$. The factorization $A(z) = A_1(z)/A_2(z)$ into stable causal all passes yields a stable causal filter $(z^{-6}A_2(z) + A_1(z))/2$ with the same magnitude response.

$K = 6$, $L = 3$, $(N = 9)$, and $d = 6$. Then $\tau = -3/2$, and the all pass $A(z) = z^{-N}D(1/z)/D(z)$ is to be formed from the flat delay filter $1/D(z)$ shown in Fig. 3. The unwrapped phase responses of z^{-d} and $A(z)$ are shown in Fig. 4. Note that the unwrapped phase of $A(z)$ takes on the value $(d-1)\pi$ at $\omega = \pi$. The low-pass behavior of their sum follows from this behavior of the phase difference. However, as $A(z)$ has two poles outside the unit circle, the factorization $A(z) = A_1(z)/A_2(z)$, where $A_1(z)$ and $A_2(z)$ are stable causal, is necessary. In this example, $A_1(z)$ is of degree $n_1 = 7$ and $A_2(z)$ is of degree $n_2 = 2$. The unwrapped phase responses of $z^{-d}A_2(z)$ and $A_1(z)$ are shown in Fig. 4, where the phase difference remains unchanged. The magnitude response, group delay, and pole–zero plot of $(z^{-d}A_2(z) + A_1(z))/2$ are shown in Fig. 5. Note that some of the poles and zeros inside the unit circle almost cancel, a behavior which has been noted before [11].

We have formed low-pass filters from flat delay filters whose group delay derivatives vanish at $\omega = 0$ and $\omega = \pi$. In the following, it is shown to be the case that the low-pass filters obtained in this way have magnitude response derivatives that vanish at $\omega = 0$ and $\omega = \pi$.

Beginning with the transfer function

$$H(z) = \frac{1}{2}(z^{-d} + A(z))$$

where $A(e^{j\omega}) = e^{j\theta(\omega)}$, the square magnitude response of $H(z)$ will be denoted by $F(\omega)$

$$F(\omega) = |H(e^{j\omega})|^2 \tag{20}$$
$$= H(e^{j\omega})H(e^{-j\omega}) \tag{21}$$
$$= \frac{1}{4}\left(e^{-jd\omega} + e^{j\theta(\omega)}\right)\left(e^{jd\omega} + e^{-j\theta(\omega)}\right) \tag{22}$$
$$= \frac{1}{2} + \frac{1}{2}\cos(d\omega + \theta(\omega)). \tag{23}$$

Consider first the point $\omega = 0$. With the flat behavior of $G(\omega)$, the group delay of $A(\omega)$ at $\omega = 0$, $G(\omega)$ can be written as $G(\omega) = d + c_0\omega^{2K} + O(\omega^{2K+1})$. Therefore $\theta(\omega) = -d\omega + c\omega^{2K+1} + O(\omega^{2K+2})$ near $\omega = 0$. Note that $1/2 + \cos(x)/2 = 1 - x^2/4 + O(x^4)$ near $x = 0$, and hence

$$F(\omega) = 1 - \frac{1}{4}c^2\omega^{4K+2} + O(\omega^{4K+3})$$

near $\omega = 0$. Therefore $F^{(i)}(\omega = 0) = 0$ for $i = 1, \cdots, 4K+1$.

Now consider the point $\omega = \pi$. With the flat behavior of $G(\omega)$ at $\omega = \pi$, $G(\omega)$ can be written as $G(\omega) = d + c_0(\omega - \pi)^{2L} + O((\omega - \pi)^{2L+1})$ near $\omega = \pi$. Therefore $\theta(\omega) = c_2 - d\,\omega + c\,(\omega - \pi)^{2L+1} + O((\omega - \pi)^{2L+2})$. Because $d\pi + \theta(\pi)$ must be an odd multiple of π (so that $F(\pi) = 0$), c_2 is equal to $l\pi$ where l is some odd integer. Note that $1/2 + \cos(x)/2 = (x - l\pi)^2/4 + O((x - l\pi)^4)$ near $x = l\pi$, hence,

$$F(\omega) = \frac{1}{4}c^2(\omega - \pi)^{4L+2} + O((\omega - \pi)^{4L+3})$$

near $\omega = \pi$. Therefore $F^{(i)}(\omega = \pi) = 0$ for $i = 0, \cdots, 4L+1$. It follows that the transfer function $H(z)$ in (18) possesses a zero at $z = -1$ of multiplicity $2L + 1$.

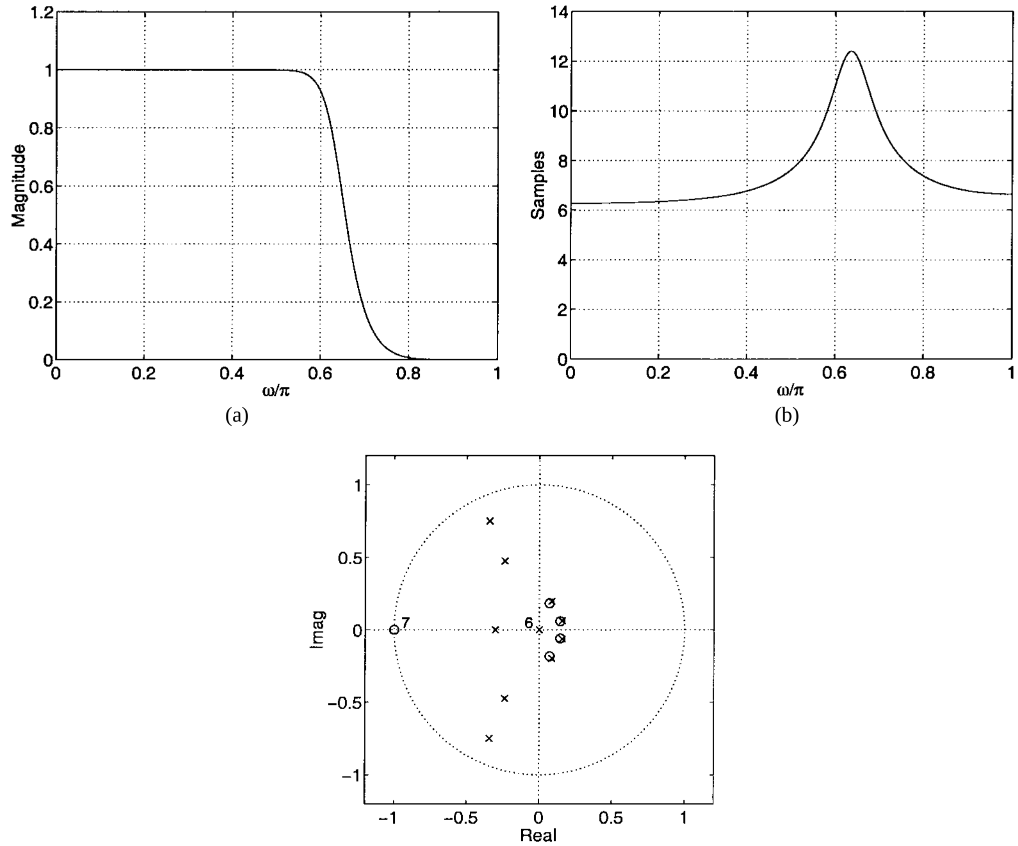
The flatness properties of $F(\omega)$ correspond to the flatness properties of $G(\omega)$. Summarizing, with $F(\omega) = |H(e^{j\omega})|^2$

$$F(\omega = 0) = 1 \tag{24}$$
$$F^{(i)}(\omega = 0) = 0 \quad \text{for } i = 1, \cdots, 4K + 1. \tag{25}$$
$$F^{(i)}(\omega = \pi) = 0 \quad \text{for } i = 0, \cdots, 4L + 1. \tag{26}$$

Example 3: By varying K and L, while keeping $N = K + L$ and d constant, the location of the cutoff frequency can be varied. For $N = 9$, $d = 8$, Fig. 6 illustrates the solutions obtained by varying K from 1 to 8.

Example 4: By varying d, while keeping K and L constant, the delay response of $H(z)$ can be varied. For $K = 5$ and $L = 4$, Fig. 7 illustrates the solutions obtained by varying d from 2 to 8 in increments of 2. Note that for larger values of d, the overall delay is greater, but that the phase is more linear. Note that the filters in Fig. 7 do not all posses the same cutoff frequency ω_c. That is to be expected because ω_c is not part of the problem formulated. The cutoff frequency is determined by the parameters K and L. However, in Section IV, it will be shown how to fine-tune ω_c so that the filters achieve prescribed values of ω_c.

The design method suffers from the drawback that there is no simple formula for selecting K, L, and d to obtain



(a) (b)

(c)

Fig. 5. Sum of two all-pass filters, $K = 6$, $L = 3$, $d = 6$. The zeros that lie outside the unit circle are not shown. Since the numerator is symmetric real, for each zero z_i inside the unit circle, another zero lies at $1/z_i$.

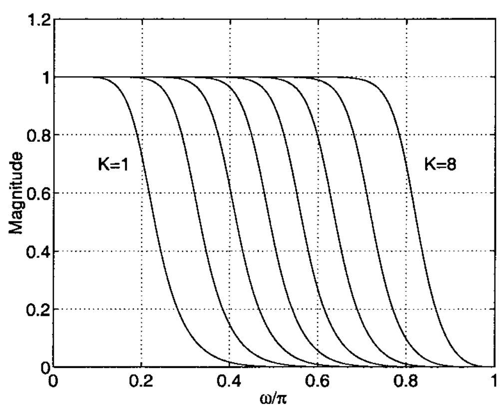


Fig. 6. Sum of two all-pass filters. $N = 9$, $d = 8$, K is varied from 1 to 8, $L = N - K$. (It follows that $\tau = -\frac{1}{2}$).

a filter having a prescribed cutoff frequency ω_c. However, the relationship between these three parameters and the filter characteristics, such as ω_c and the DC group delay $G(0)$, can be investigated by plotting $(\omega_c, G(0))$ as a coordinate in a plane. In this way, the qualitative behavior of ω_c and $G(0)$ as a function of K, L, and d can be examined.

For $N = K + L = 9$, Fig. 8 illustrates this for various low-pass filters $H(z)$ obtained from degree 9 allpole filters. The parameters (d, K, L) are indicated in the figure for $N = K + L = 9$; the frequency ω_c is that frequency at which $|H(e^{j\omega})| = 1/2$.

IV. Continuous Variation of the Cutoff Frequency

For a fixed value of d, the design approach described above produces a discrete set of d filters. For these filters, the location of the cutoff frequency is entirely determined by d, K, and L. However, by giving up a single derivative constraint, the location of the cutoff frequency can be continuously varied. In this way, the cutoff frequency can be precisely specified, if desired. The method begins with the flat delay filter described in Section II. Let us make the dependence of a_n on K and L in (5) explicit by the notation $a_n(K, L)$. Similarly, we will use the notation $D(z; K, L)$, etc. The ability to vary the cutoff frequency is obtained by forming a weighted average of two filters

$$D(z; K, L, \alpha) = \alpha \cdot D(z; K, L+1) + (1-\alpha) \cdot D(z; K+1, L) \quad (27)$$

$$= \sum_{n=0}^{K+L+1} (\alpha \cdot a_n(K, L+1) + (1-\alpha) \cdot a_n(K+1, L)) z^{-1} \quad (28)$$

where α lies in the real interval $[0, 1]$. The group delay of $1/D(z; K, L, \alpha)$ has a flat characteristic at $\omega = 0$ and $\omega = \pi$, but the total number of contiguous vanishing derivatives is one less than the maximum number achievable. The reason this *linear* combination of coefficients retains the flatness parameters K and L, is that (3), (4) are linear equations. Note that $D(z; K, L, \alpha)$ is of degree $K + L + 1$.

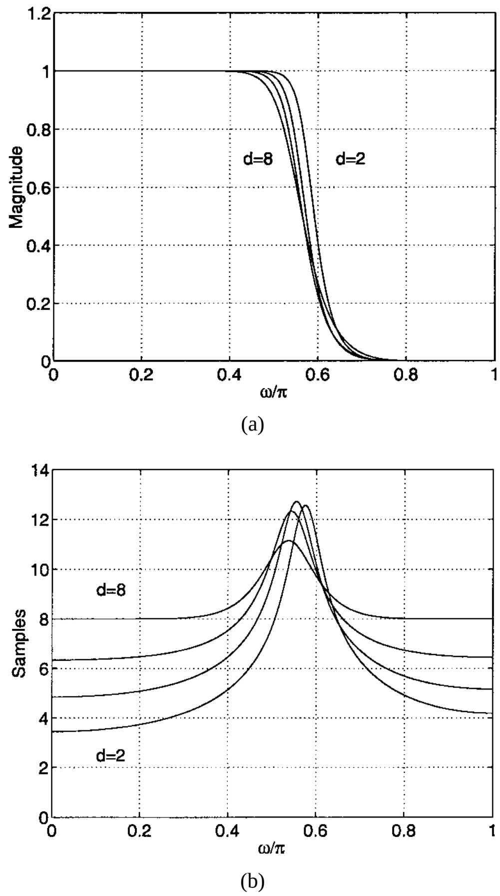


Fig. 7. Sum of two all-pass filters, $K = 5$, $L = 4$, d is varied from 2 to 8 in increments of two. (It follows that τ is varied from -3.5 to -0.5 in increments of 1.)

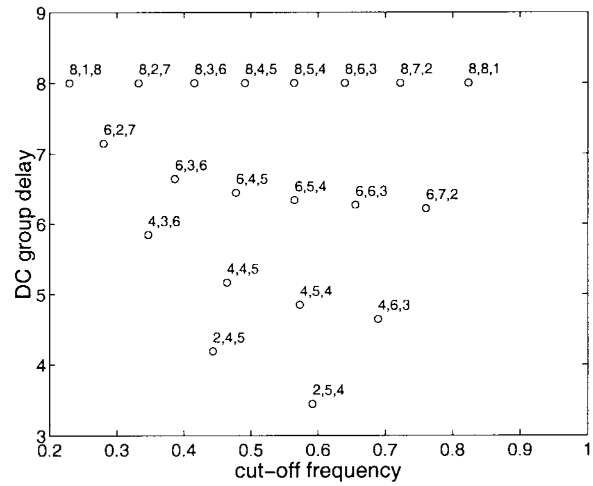


Fig. 8. Locations in the ω_c-$G(0)$ plane of maximally flat all-pass sums (8) for which the sum of the degrees of $A_1(z)$ and $A_2(z)$ is 9. The parameters (d, K, L) are indicated in the figure.

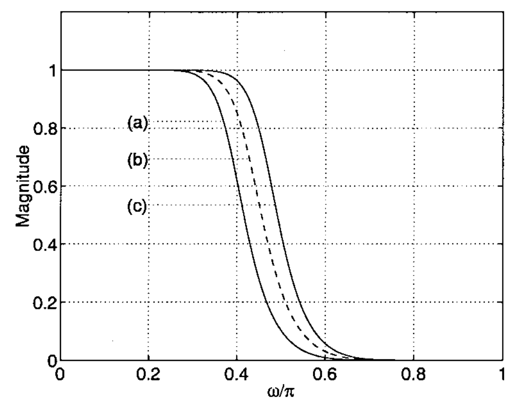


Fig. 9. Variation of cutoff frequency. For each of the filters shown, $d = 8$, and N, the degree of the all-pass filter $A(z)$ is nine. The flatness parameters of the three filters are: (a) $K = 3$, $L = 6$, (b) $K = 3$, $L = 5$, (c) $K = 4$, $L = 5$. The filter shown as a dashed line, (b), is obtained by using a weighted average as described in the text with $\alpha = 0.5$.

By continuously varying α in $[0, 1]$, the cutoff frequency can be continuously varied. The all pass $A(z)$ and the low pass $H(z)$ are now written as

$$A(z; K, L, \alpha) = \frac{z^{-(K+L+1)} D(1/z; K, L, \alpha)}{D(z; K, L, \alpha)} \tag{29}$$

$$H(z; \alpha) = \frac{1}{2}(z^{-d} + A(z; K, L, \alpha)). \tag{30}$$

Certainly there is no simple functional form for the way in which the zeros vary with α. Therefore, the formation of $D_1(z)$ and $D_2(z)$ for a specific value of α requires computing the zeros of $D(z; K, L, \alpha)$ for that value.

When a weighted average is used, as described in this section, the conditions which d, K, and L must satisfy, given in Section III, are to be modified. Because one of the derivative conditions is given up, the conditions become somewhat looser. The less restrictive conditions on d, K, and L are:

1) $|K - L| \leq d \leq K + L + 2$;
2) d must have the same parity as $K + L$.

Note that when K and L are chosen to be equal, d may be selected to be zero. In this case, the weighted average filter is the classical digital Butterworth filter of odd degree. Hence, the filters described in this paper can be interpreted as a generalization of the Butterworth filter. Note that the generalization of the Butterworth filter described in this paper is quite different from the generalization described in [39]. The filters described in this paper are realizable as all-pass sums, while those in [39] did not have that constraint imposed.

Example 5: As an example, with $K = 3$, $L = 5$, and $d = 8$, we obtain a degree 9 all-pass filter. For $\alpha = 0.5$, the magnitude response of the all-pass sum is shown as a dashed line in Fig. 9.

Fig. 10 illustrates some of the achievable points in the ω_c-$G(0)$ plane when $A(z)$ is of degree 9, obtained by continuously varying the cutoff frequency ω_c. This figure indicates how to choose the parameters (d, K, L) to obtain a filter having a desired bandwidth and delay. It is interesting to note that at the vertices, the curves are slightly cusped.

How does one choose α to satisfy a prescribed cutoff frequency ω_c? First, one should check that the desired cutoff frequency ω_c lies between the cutoff frequencies of $H(z; \alpha = 0)$ and $H(z; \alpha = 1)$. Otherwise, the following formula for α cannot be expected to yield a meaningful solution. Given ω_c, the method to calculate the appropriate value α can be

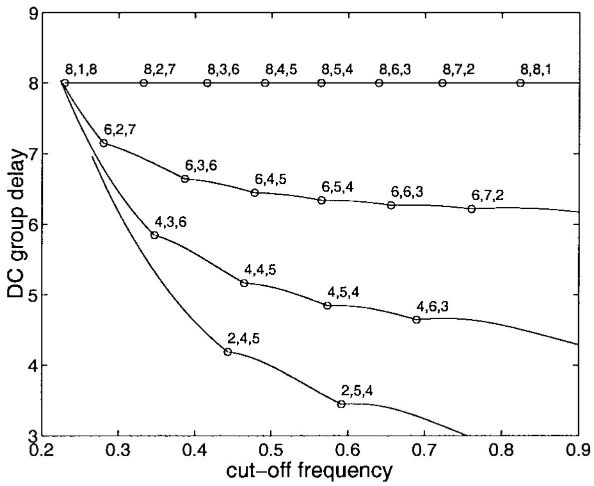


Fig. 10. Locations in the ω_c-$G(0)$ plane of all-pass sums (8) for which the sum of the degrees of $A_1(z)$ and $A_2(z)$ is nine, obtained by the continuous variation of the the cutoff frequency ω_c.

derived by beginning with the equation $|H(e^{j\omega_c})| = 1/2$. Using the expression for the square magnitude (23), one then gets $1/2 + 1/2\cos(d\omega_c + \theta(\omega_c)) = 1/4$ where $A(e^{j\omega}) = e^{j\theta(\omega)}$. Using (11) one obtains $\theta(\omega) = -2\beta(\omega) - N\omega$ where $D(e^{j\omega}) = |D(e^{j\omega})|e^{j\beta(\omega)}$. In turn, this gives

$$\frac{1}{2} + \frac{1}{2}\cos(d\omega_c - 2\beta(\omega_c) - N\omega_c) = \frac{1}{4}.$$

Solving for $\beta(\omega_c)$, the desired value of the phase of $D(z; K, L, \alpha)$ at $z = e^{j\omega_c}$, gives

$$\beta(\omega_c) = \pm\frac{\pi}{3} - \frac{\omega_c}{2} + l\pi \tag{31}$$

where l is some integer. From the desired phase $\beta(\omega_c)$, the desired value of α can be determined. Let us define r_0, r_1, β_0, β_1 by

$$D(e^{j\omega_c}; K+1, L) =: r_0 e^{j\beta_0} \tag{32}$$

$$D(e^{j\omega_c}; K, L+1) =: r_1 e^{j\beta_1}. \tag{33}$$

Then $D(z = e^{j\omega_c}; K, L, \alpha)$ can be written as $\alpha r_1 e^{j\beta_1} + (1 - \alpha) r_0 e^{j\beta_0}$. The sought value of α is such that

$$\alpha r_1 e^{j\beta_1} + (1-\alpha) r_0 e^{j\beta_0} = X \cdot e^{j\beta_c}$$

where X is some positive real number and β_c is the desired phase (31). After some geometry, the solution is found to be

$$\alpha = \frac{r_0 \sin(\beta_0 - \beta_1) - r_0\cos(\beta_0 - \beta_1)\tan(\beta_c - \beta_1)}{r_0\sin(\beta_0 - \beta_1) + (r_1 - r_0\cos(\beta_0 - \beta_1))\tan(\beta_c - \beta_1)}$$

where $\beta_c = \pm\pi/3 - \omega_c/2$. The additive multiple of π in (31) can be dropped because tangent is periodic with π. This leaves two candidate values of α. It was found in practice that only one of those two values will lie in $[0, 1]$.

Example 6: Fig. 11 illustrates a set of low-pass filters where the ω_c is prescribed to be 0.56π. Compare to *Example 4*. The filter with least delay is the classical Butterworth filter. The figure shows that with this approach, the tradeoff between the delay and the phase linearity can be managed with out affecting the magnitude response significantly.

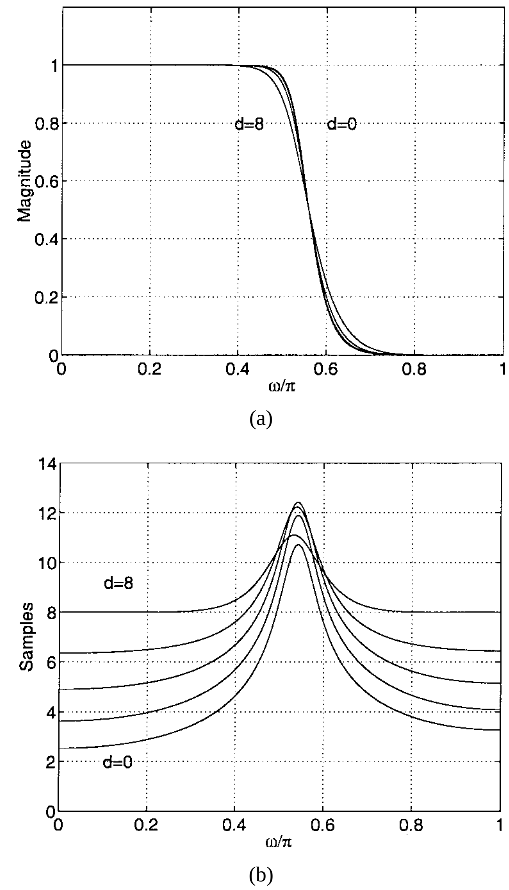


Fig. 11. Sum of two all-pass filters, with prescribed cutoff frequency $\omega_c = 0.56\pi$. $K = 4$, $L = 4$, $N = 9$, d is varied from zero to eight in increments of two. (It follows that τ is varied from -4.5 to -0.5 in increments of one.)

V. Conclusion

This paper presents two main results: 1) the design and closed-form solution of an allpole filter (or all-pass filter), the group delay of which approximates any value with prescribed flatness K and L at $\omega = 0$ and $\omega = \pi$, respectively; and 2) the application of this result to the design of low-pass filters realizable as all-pass sums, whose magnitude response is flat at $\omega = 0$ and $\omega = \pi$, accordingly. This new class of maximally flat low-pass recursive digital filters can be made to have approximately linear phase in the passband (for $d = N - 1$), and the tradeoff between delay and phase-linearity can be controlled by appropriate selection of the delay parameter d. In addition, the continuous variation of the cutoff frequency was described, in which case, for special values of K, L, and d ($K = L$, $d = 0$), the classical Butterworth digital filter of odd degree is retrieved. Note that, with the classical Butterworth filter of degree N, it is not possible to adjust the delay (or phase-linearity). However, with the more general class of filters described in this paper, the adjustment of the delay becomes possible, and the tradeoff between the delay and the phase-linearity can be chosen. Matlab programs for the the construction of the low-pass filters described in this paper, and the new maximally flat delay allpole filter on which it depends, are also provided. Additional programs are available at URL http://taco.poly.edu/selesi/.

Appendix A
The Zeros of $D(z)$

As noted in Section III, for the design of low-pass filters realizable as all-pass sums, τ will be of the form $(d-N)/2$. It is therefore useful to examine the properties of $D(z)$ when τ is an integer multiple of one half. Of particular interest is the location of the zeros of $D(z)$ with respect to the unit circle $|z|=1$. While poles of $1/D(z)$ outside the unit circle do not pose a problem [recall the factorization as in (9)], zeros *on* the unit circle will not give rise to useful solutions. Because $d \geq 0$, it is not necessary to consider integer multiples of one half that are more negative than $-N/2$.

It was found empirically that for $\tau \geq 0$, all the zeros of $D(z)$ lie inside the unit circle. The remaining values of τ to be considered are $-i/2$ for $i = 1, \cdots, N$. Let $q = 2\min(K, L) + 1$. Three ranges of i will be considered: $1 \leq i \leq q-1$, $i = q$, and $q+1 \leq i \leq N$. The following observations were made, based on numerical observation.

1) $\tau = -i/2$ with $1 \leq i \leq q-1$. For these values of τ, $2\lfloor (i+1)/4 \rfloor$ zeros of $D(z)$ lie outside the unit circle. In addition, if $i = 2 \bmod 4$, then two zeros lie on the unit circle.
2) $\tau = -q/2$. For this value of τ, $D(z) = (1+z^{-1})^q z^{-N+q}$.
3) $\tau = -i/2$ with $q+1 \leq i \leq N$. For i even, $i/2-1$ zeros of $D(z)$ lie outside the unit circle and two zeros lie on the unit circle. For i odd, $i-q$ zeros of $D(z)$ lie outside the unit circle and $i+q-1$ zeros lie at $z=-1$.

Note that, of the negative values of τ considered, only those solutions for which $1 \leq i \leq q-1$ and $i \neq 2 \bmod 4$, are free from zeros on the unit circle.

Appendix B
Restrictions

As noted in Section III, some restrictions apply to the selection of d, K, and L. Although, as stated above, the group delay of $A(z)$ in (10) should approximate d in both the passband and the stopband, this alone is not sufficient to ensure that the all-pass sum (10) is low pass. It is also necessary that $A(z) \approx -z^{-d}$ near $z=-1$. That means that for the all-pass sum (10), $A(-1)$ should be $-(-1)^d$. On the other hand, (11) immediately yields $A(-1) = (-1)^N$. It follows that d must have the same parity as $N+1$. Note that, in this case, $d-N$ is odd; therefore, the relevant values of τ, namely $(d-N)/2$, are of the form $i/2$ where i is an odd integer.

Two more conditions, in addition to the parity condition, are necessary. Recall that in Appendix A it was noted that, for $\tau = -i/2$ with $q \leq i \leq N$, the solution (5) possesses zeros on the unit circle. Since this precludes the utility of (5), we need only consider $i \leq q-1$. It follows that d must satisfy the inequality $N - d \leq q-1$; that is, $K + L - 2\min(K, L) \leq d$, or equivalently, $|K-L| \leq d$. In combination with the parity condition, the condition $|K-L|+1 \leq d$ follows.

The last restriction is $d \leq N+1$. For values of d greater than this, the unwrapped phase of $A(z)$ differs from the unwrapped phase of z^{-d} by π in $(0, \pi)$. Consequently, the magnitude response of the all-pass sum will posses zeros in $(0, \pi)$, and points of unity gain between these zeros and $\omega = \pi$. Therefore, the magnitude response when $d > N+1$ is hardly low pass.

Taken together, these considerations yield the requirements $|K-L|+1 \leq d \leq K+L+1$ and d must have the same parity as $K+L+1$.

Appendix C
Programs

A. Calculating the Flat Delay Filter

Although (5) appears rather cumbersome, the coefficients a_n are very simple to compute because a recursive relationship exists. Let us begin with (6). Note that in (6)

$$\frac{a_{n+1}}{a_n} = -\frac{\binom{N}{n+1}}{\binom{N}{n}} \cdot \frac{(2\tau)_{n+1}(2\tau+N+1)_n}{(2\tau)_n(2\tau+N+1)_{n+1}} \tag{34}$$

$$= -\frac{N-n}{n+1} \cdot \frac{2\tau+n}{2\tau+N+1+n}. \tag{35}$$

With $a_0 = 1$, and the ratio (35), a_{n+1} can be computed from a_n by simply multiplying a_n by this ratio. The Matlab command for the cumulative product **cumprod** makes it possible to write (6) in two lines, shown at the bottom of the page.

The procedure for computing a_n in (5) is similar. Write a_n as $a_n = \sum_{i=0}^{L} c_{i,n}$ where

$$c_{i,n} = \frac{(-4)^i(-1)^n(\tau)_i(n-i+1)_i(2\tau+2i)_{n-i}}{(2\tau+K+L+1)_n(K+L+1-i)_i} \times \binom{K+L}{n}\binom{L}{i}. \tag{36}$$

Then

$$\frac{c_{i,n+1}}{c_{i,n}} = -\frac{K+L-n}{n-i+1} \cdot \frac{2\tau+n+i}{2\tau+K+L+1+n}. \tag{37}$$

Therefore, $c_{i,n+1}$ can be computed by multiplying $c_{i,n}$ by this ratio. Note, however, that $c_{i,n}$ is zero for $n < i$, so the cumulative product operation must begin with the first nonzero term $c_{i,i}$. It turns out that these terms can also be computed by the cumulative product operation; the ratio $c_{i+1,i+1}/c_{i,i}$ is also a simple rational function. The simple Matlab program in Table II is obtained by putting these ratios together. The variable **c** denotes the $c_{i,i}$ term, and it is updated as **i** is incremented.

B. Calculating the All-Pass Sum

The program **mfaps**, for "maximally flat all-pass sum," in Table III, implements the design procedure described in this paper. Given the flatness parameters K and L, and the delay d in (8), the program returns the all-pass filters $A_1(z)$ and $A_2(z)$, so that the all-pass sum (8) is low pass. The sum of the

```
n = 0 : N - 1;
a = cumprod([1, (n - N). * (2 * t + n)./(n + 1)./(2 * t + N + 1 + n)])
```

TABLE II
MATLAB PROGRAM FOR THE FLAT DELAY FILTER

```
function [b,a] = flatdelay(K,L,t)
% K, L : number of conditions at w=0, w=pi
% t : group delay
% b/a : digital allpole filter of degree K+L
% [b,a] = flatdelay(6,4,3.2); % example

N = K+L;
a = zeros(1,N+1);
c = 1;
for i = 0:L
   n = i:N-1;
   v = (n-N) ./ (n-i+1) .* (2*t+n+i) ./ (2*t+N+1+n);
   a = a + [zeros(1,i), cumprod([c, v])];
   c = c * 4 * (t+i) * (L-i) / (2*t+N+1+i) / (i+1);
end
b = sum(a);
```

TABLE III
MATLAB PROGRAM FOR THE DESIGN OF A
LOW-PASS FILTER REALIZABLE AS AN ALL-PASS SUM

```
function [a1,a2,p,q] = mfaps(K,L,d)
% Design of a maximally flat lowpass filter H(z) as the
% sum of two allpass filters: H(z) = z^(-d) A2(z) + A1(z).
% K, L : number of conditions at w=0, w=pi
% Note: two conditions must be satisfied
%   (1) abs(K-L)+1 <= d <= K+L+1
%   (2) d must be same parity as K+L+1
% a1, a2 : the denominators of the allpass filters A1(z), A2(z)
% p/q : overall transfer function

% check input for validity:
b1 = (abs(K-L)+1 <= d) & (d <= K+L+1);
b2 = rem(K+L+1-d,2)==0;
if ~(b1 & b2)
   disp('For this K and L, d must be one of the following:');
   disp((abs(K-L)+1):2:(K+L+1));
   break
end

[tmp,a] = flatdelay(K,L,(d-K-L)/2);
rts = roots(a);
v = abs(rts)<1;
a1 = real(poly(rts(v)));        % roots inside unit circle
a2 = real(poly(1./rts(~v)));    % roots outside unit circle

% compute overall transfer function p/q
p = [zeros(1,d), a] + [a(K+L+1:-1:1) zeros(1,d)];
q = conv(a1,a2);
p = p*sum(q)/sum(p);            % normalize
```

degrees of $A_1(z)$ and $A_2(z)$ is $K+L$. The frequency response magnitude has a flat characteristic at $\omega = 0$ and $\omega = \pi$. The all-pass filters are represented by their denominators $\mathbf{a1}$ and $\mathbf{a2}$. The program also gives $\mathbf{p}$ and $\mathbf{q}$ which represent the total low-pass response.

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