Narrow-Band Low-Pass Digital Differentiator Design

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Ideal Lowpass Digital Differentiator

The frequency response of the ideal lowpass digital differentiator is

$$H_{LP}(e^{j\omega}) = \begin{cases} j\,\omega & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| < \pi \end{cases} \tag{1}$$

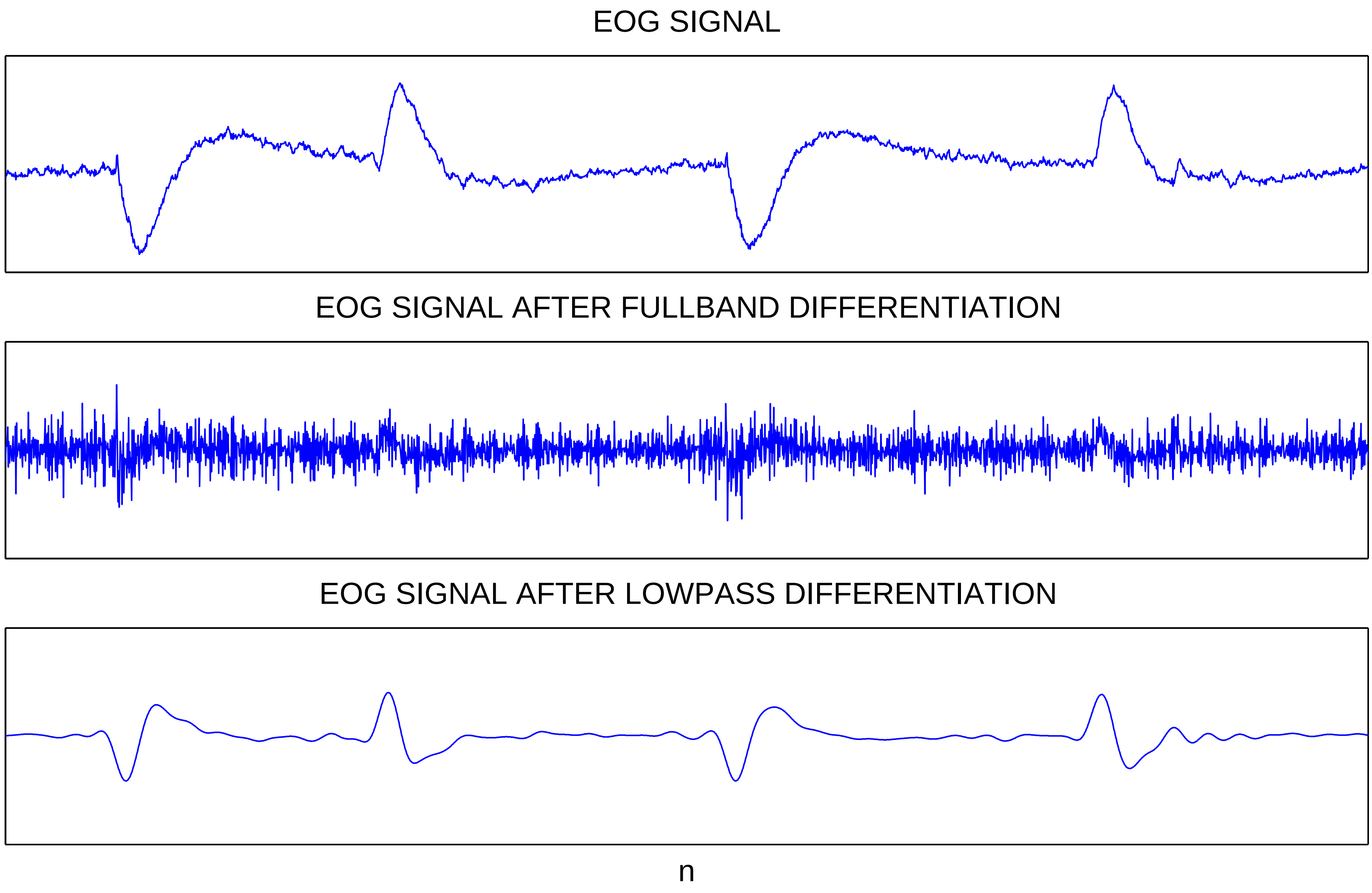
- It is a narrow-band filter if ω_c is much smaller than π.
- A narrow-band filter should have a long impulse response.
- $\Longrightarrow$ It is desirable to have simple design algorithms so that ill-conditioning and computational complexity is minimized.
- The window method for FIR filter design is a natural choice in this case. The design method described here gives an alternative approach.

EOG Example

The next slide illustrates the result of filtering an EOG signal with:

1. a full-band differentiator and
2. a narrow-band lowpass differentiator

Differentiation with the full-band differentiator yields an extremely noisy signal, while lowpass differentiation gives a more useful result.


EOG SIGNAL
EOG SIGNAL AFTER FULLBAND DIFFERENTIATION
EOG SIGNAL AFTER LOWPASS DIFFERENTIATION
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Design

To avoid the undesirable amplification of noise in digital differentiation, lowpass differentiators can be used in place of full-band ones.

Low-pass digital differentiator design:

1. Maxflat
2. Least-squares
3. Remez
4. Flat passband, Equiripple stopbands (Kaiser, Rabiner, Vaidyanathan)

Our Approach

We describes a simple formulation for the non-iterative design of narrow-band FIR linear-phase lowpass digital differentiators.

- The filters are flat around dc and have equally spaced nulls in the stopband.
- The impulse response can be written as a sum of sines (Frequency sampling expression).
- The design problem is formulated so as to avoid the complexity or ill-conditioning of standard methods for the design of similar filters when those methods are used to design narrow-band filters with long impulse responses.

The sinc function is given by

$$\operatorname{sinc}(f) := \frac{\sin(\pi\, f)}{\pi\, f}.$$

The function $\operatorname{sinc}(f)$ is symmetric ($\operatorname{sinc}(-f) = \operatorname{sinc}(f)$) and equal to zero for $f = \pm 1, \pm 2, \pm 3, \ldots$; therefore if we define

$$s_k(f) := \operatorname{sinc}(f-k) - \operatorname{sinc}(f+k)$$

then we have

1. $s_k(f)$ is antisymmetric, $\operatorname{sinc}(-f) = -\operatorname{sinc}(f)$.
2. $s_k(f) = 0$ for $f \in \mathbb{Z}/\{\pm k\}$.
   ($s_k(f) = 0$ whenever f is an integer different from $\pm k$.)

Analog Lowpass Differentiator

The digital filter design procedure we propose begins with an analog frequency response having the following form:

$$A(f) = \sum_{k=1}^{K} a(k,K)\ (\mathrm{sinc}(f-k) - \mathrm{sinc}(f+k))$$

Therefore, the frequency response $A(f)$ has the following properties:

1. $A(f)$ is antisymmetric, $A(-f) = -A(f)$.

2. $A(f) = 0$ for $f = 0$, and for $f = \pm(K+1), \pm(K+2), \pm(K+3), \ldots$.

- The frequency response $A(f)$ is zero at $f = 0$.
- The first null in the stopband depends on K.
- The exact behavior of $A(f)$ depends on the coefficients $a(k,K)$, however, the uniformly spaced nulls in the stopband ensures that the attenuation increases with frequency.

The coefficients $a(k, K)$ are to be determined so that the frequency response $A(f)$ approximates f near $f = 0$

$$A(f) \approx f.$$

Given K, find $a(k, K)$ for $1 \leq k \leq K$ such that the derivatives of $A(f)$ at $f = 0$ match the derivatives of the ideal differentiator $\mathrm{IdealDiff}(f) := f$ at the point $f = 0$:

$$A^{(1)}(0) = 1 \tag{2}$$

$$A^{(i)}(0) = 0, \qquad i = 3, 5, \ldots, 2\,K - 1. \tag{3}$$

- The even derivatives are automatically zero because $A(f)$ is an odd function, $A(-f) = -A(f)$.
- This is a linear system of equations with an equal number of equations and variables.
- The stopband of $A(f)$ is neither equiripple nor maximally flat.

Example

For example, when $K = 1$, we have

$$a(1,1) = \frac{1}{2}.$$

When $K = 2$, we have

$$a(1,2) = -\frac{1}{6} + \frac{1}{9}\pi^2$$
$$a(2,2) = -\frac{4}{3} + \frac{2}{9}\pi^2$$

When $K = 3$, we have

$$a(1,3) = \frac{1}{48} - \frac{13}{288}\pi^2 + \frac{7}{480}\pi^4$$
$$a(2,3) = \frac{16}{15} - \frac{16}{9}\pi^2 + \frac{14}{75}\pi^4$$
$$a(3,3) = \frac{243}{80} - \frac{81}{32}\pi^2 + \frac{189}{800}\pi^4$$

Example

When $K = 4$, we have

$$a(1,4) = -\frac{1}{720} + \frac{29}{4320}\pi^2 - \frac{427}{64800}\pi^4 + \frac{31}{18900}\pi^6$$
$$a(2,4) = -\frac{16}{45} + \frac{208}{135}\pi^2 - \frac{2366}{2025}\pi^4 + \frac{496}{4725}\pi^6$$
$$a(3,4) = -\frac{2187}{560} + \frac{2187}{160}\pi^2 - \frac{5103}{800}\pi^4 + \frac{2511}{4900}\pi^6$$
$$a(4,4) = -\frac{2048}{315} + \frac{2048}{135}\pi^2 - \frac{12544}{2025}\pi^4 + \frac{15872}{33075}\pi^6$$

Other values $a(k, K)$ can be easily computed.

Conversion to Digital Filter

To convert the analog frequency response $A(f)$ into a digital frequency response $D(f)$, we can use the digital sinc function in place of the usual sinc function.

The digital sinc function $\mathrm{dsinc}(f, N)$ can be written as

$$\mathrm{dsinc}(f, N) := \frac{\sin(N\pi\, f)}{\sin(\pi\, f)}. \tag{4}$$

- The digital sinc function defined in (4) is periodic in f with period 2:

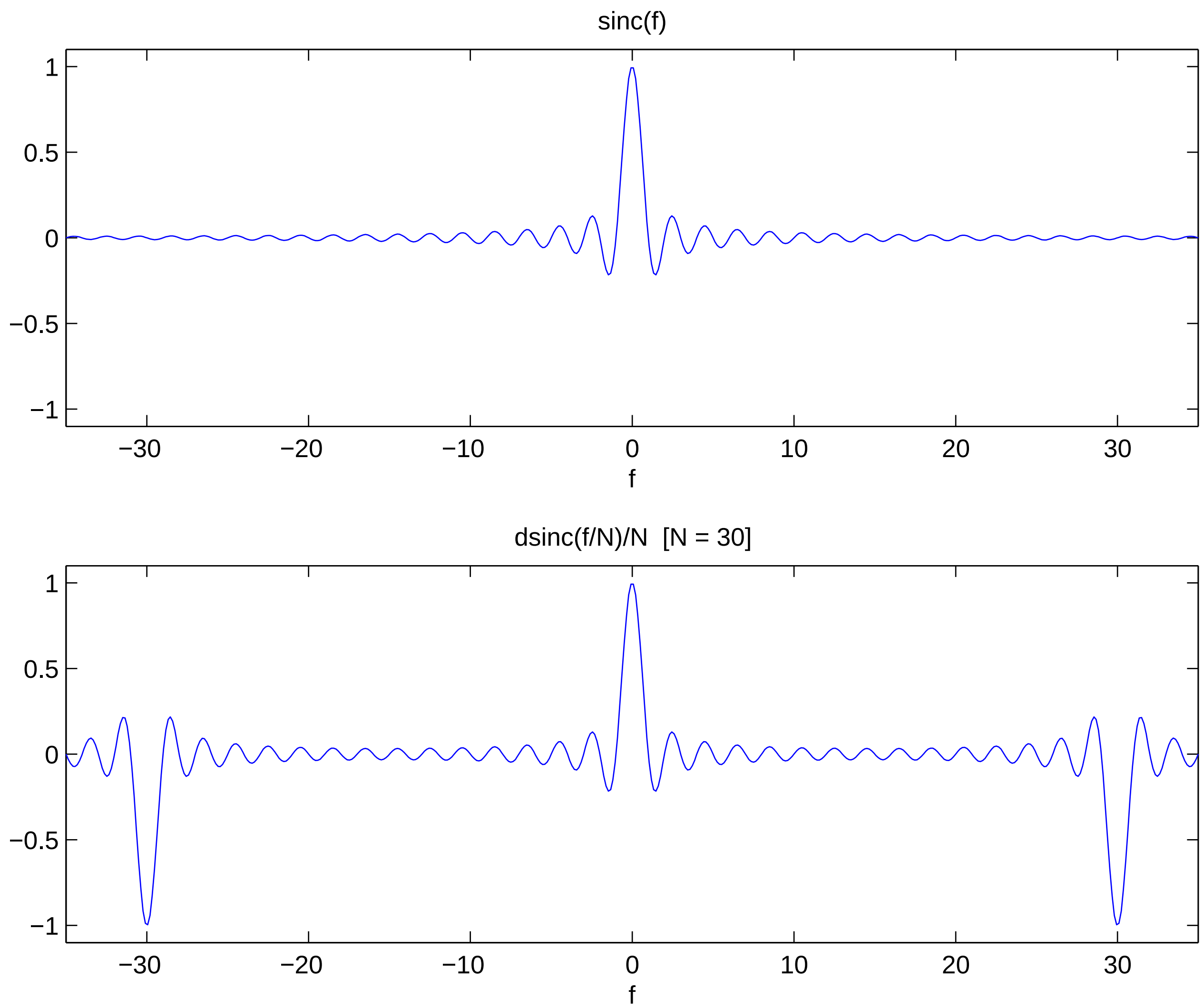
$$\mathrm{dsinc}(f + 2) = \mathrm{dsinc}(f).$$

- We have the following approximation:

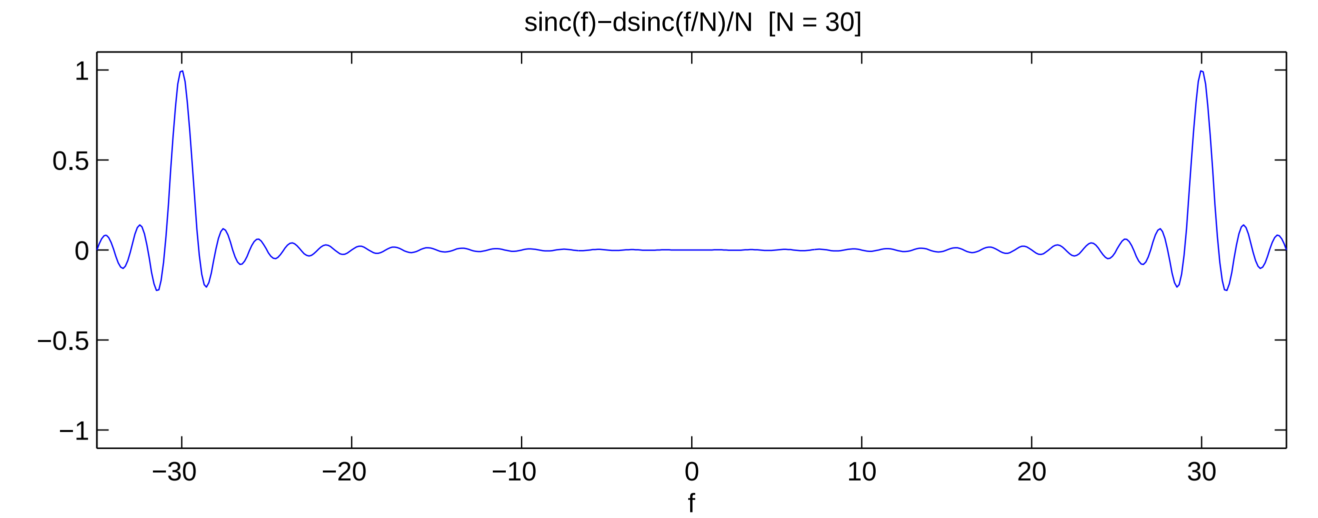
$$\mathrm{sinc}(f) \approx \frac{1}{N}\,\mathrm{dsinc}\left(\frac{f}{N}, N\right) \quad \text{for} \quad |f| < 0.5\,N.$$

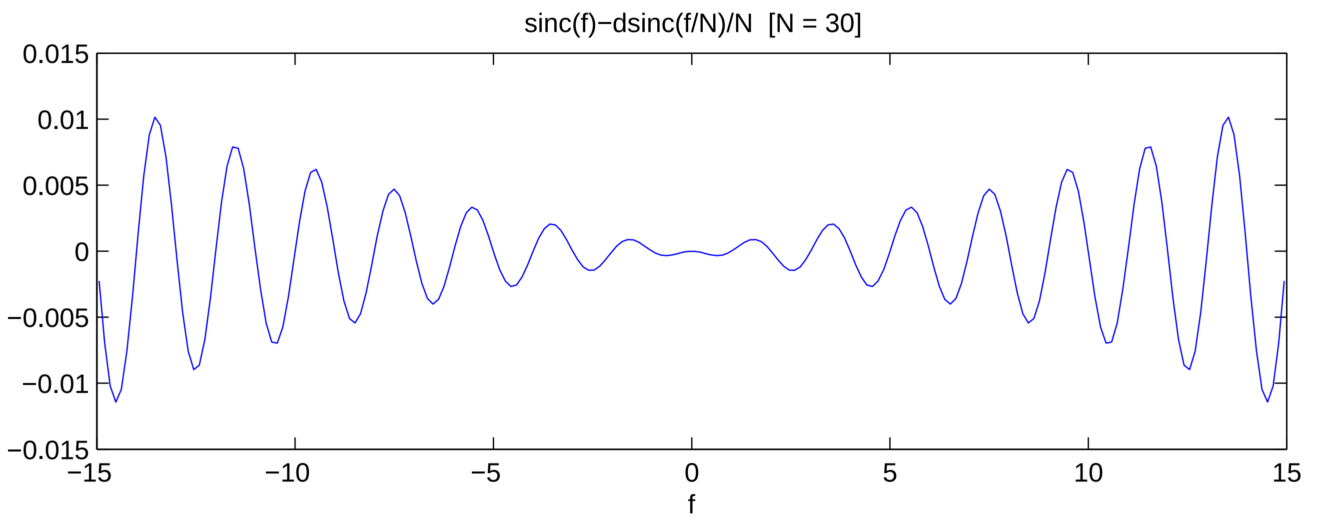
  for large values of N.

Sinc vs. Digital Sinc



Sinc Minus Digital Sinc





Sinc vs. Digital Sinc

$$\mathrm{sinc}(f) \approx \mathrm{dsinc}(f/N)/N, \quad \text{for} \quad |f| < 0.5N$$

especially for large values of N.

The design of digital differentiators described here is intended for long impulse responses. In this case, N is large and the approximation is valid.

Digital Lowpass Differentiators

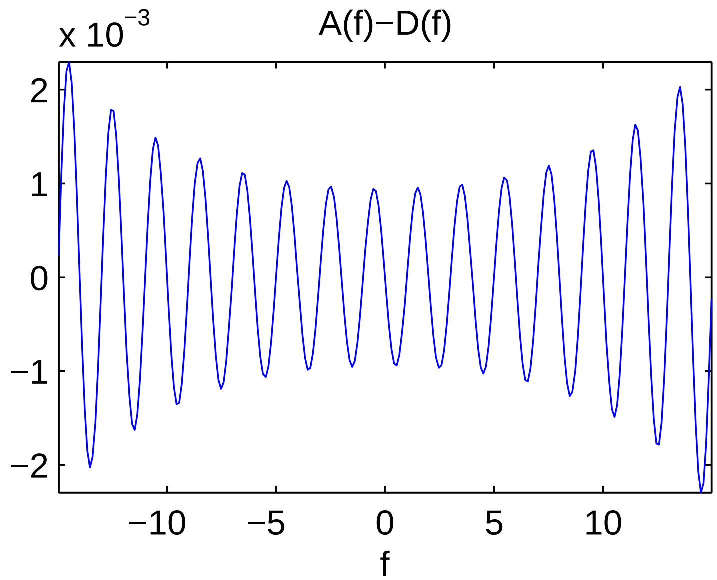
Consider the function $D(f)$, based on the digital sinc function:

$$D(f) = \frac{1}{N} \sum_{k=1}^{K} a(k, K) \left[\mathrm{dsinc}\left(\frac{f-k}{N}, N \right) - \mathrm{dsinc}\left(\frac{f+k}{N}, N \right) \right]$$

For $N > K$, we have the approximation

$$D(f) \approx A(f) \quad \text{for} \quad |f| < N/2.$$

For example: with $K = 3$, $N = 30$:



Digital Lowpass Differentiators

Consider the function $D(f)$, based on the digital sinc function:

$$D(f) = \frac{1}{N} \sum_{k=1}^{K} a(k, K) \left[\mathrm{dsinc}\left(\frac{f-k}{N}, N \right) - \mathrm{dsinc}\left(\frac{f+k}{N}, N \right) \right]$$

- The function $D(\frac{N}{2\pi}\omega)$ is then a 2π periodic function of ω and can therefore be used as the frequency response $H(e^{j\omega})$ of a digital filter.
- Then $H(e^{j\omega})$ will be approximately maximally flat at $\omega = 0$.

The impulse response $h(n)$ is given by the inverse discrete-time Fourier transform of $H(e^{j\omega})$,

$$h(n) = \mathrm{IDTFT}\left\{ e^{-j(\frac{N-1}{2})\omega} \cdot D\left(\frac{N}{2\,\pi}\,\omega\right)\right\}$$

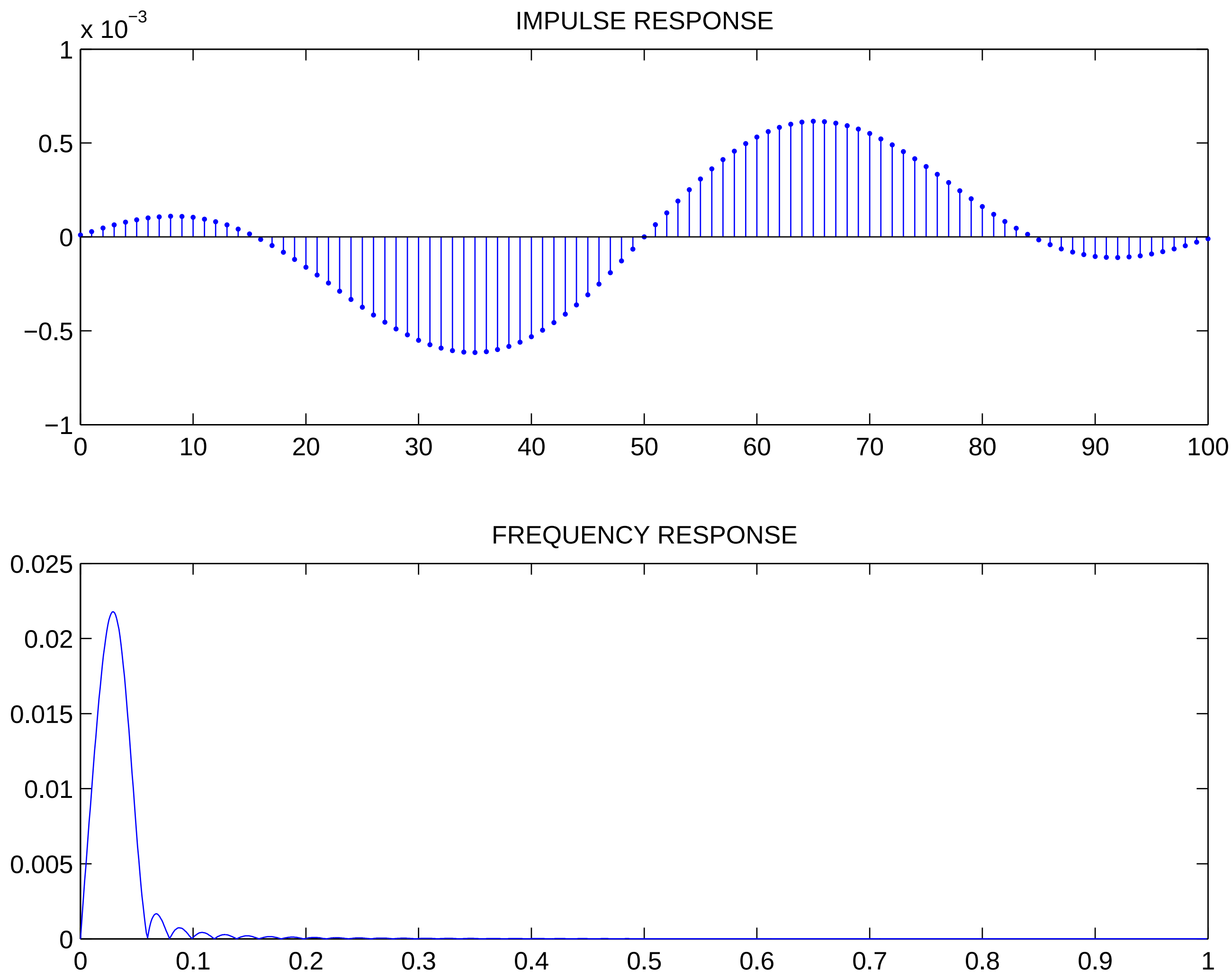
where the phase term is included to make $h(n)$ causal. Then $h(n)$ is a linear-phase FIR impulse response of length N:

$$h(n) = \frac{1}{N^2}\sum_{k=1}^{K} a(k,K)\,\sin\left(\frac{2\pi k}{N}\left(n - \frac{(N-1)}{2}\right)\right)$$

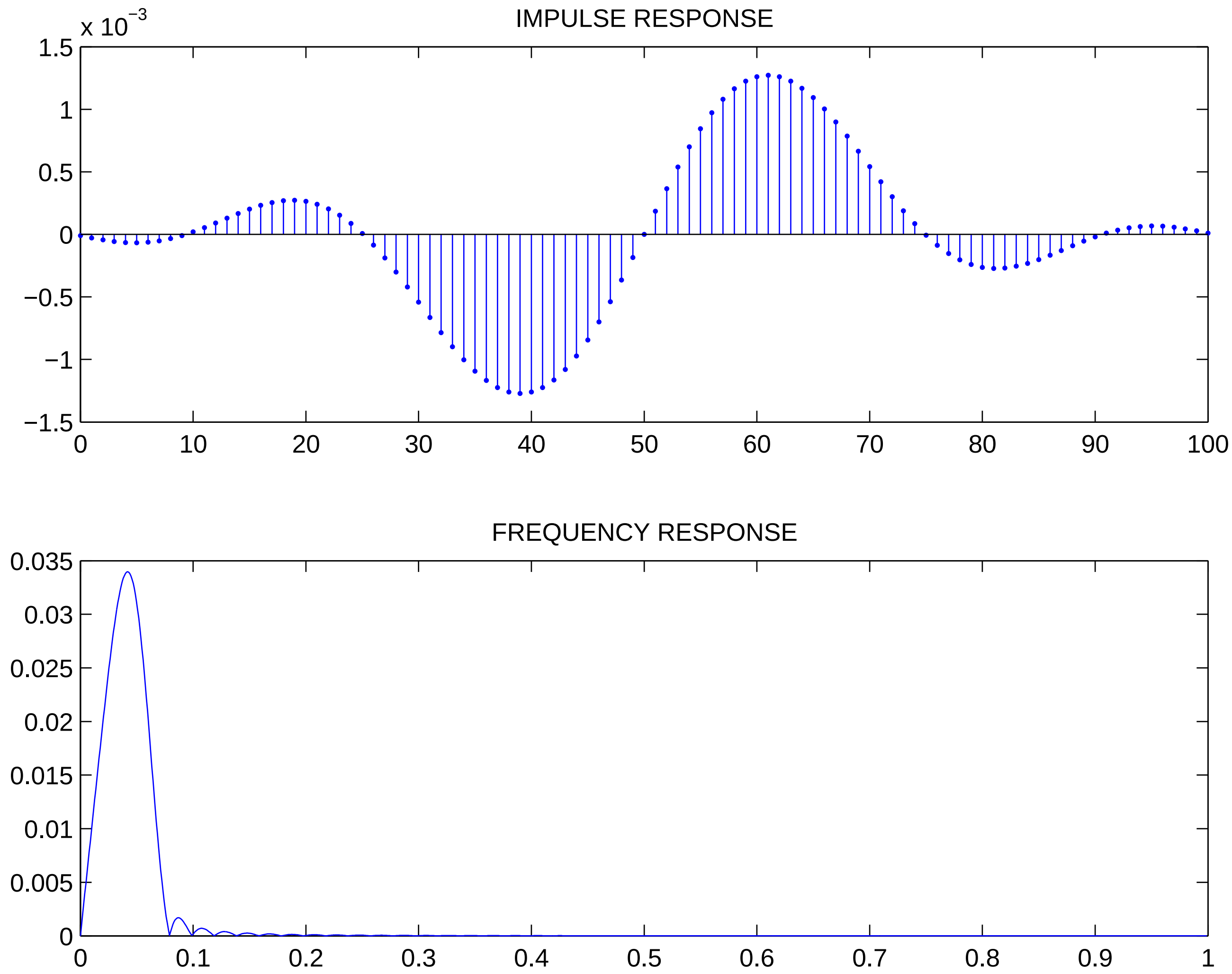
for $0 \le n \le N-1$. Once a table of values $a(k,K)$ is computed, it can be used regardless of the length N of the impulse response $h(n)$.

The width of the passband is controlled by the parameter K, the number of flatness constraints at dc. Note that $2\,(K-1)$ is the number of zeros of $H(z)$ that lie away from the unit circle, as illustrated in the following examples.

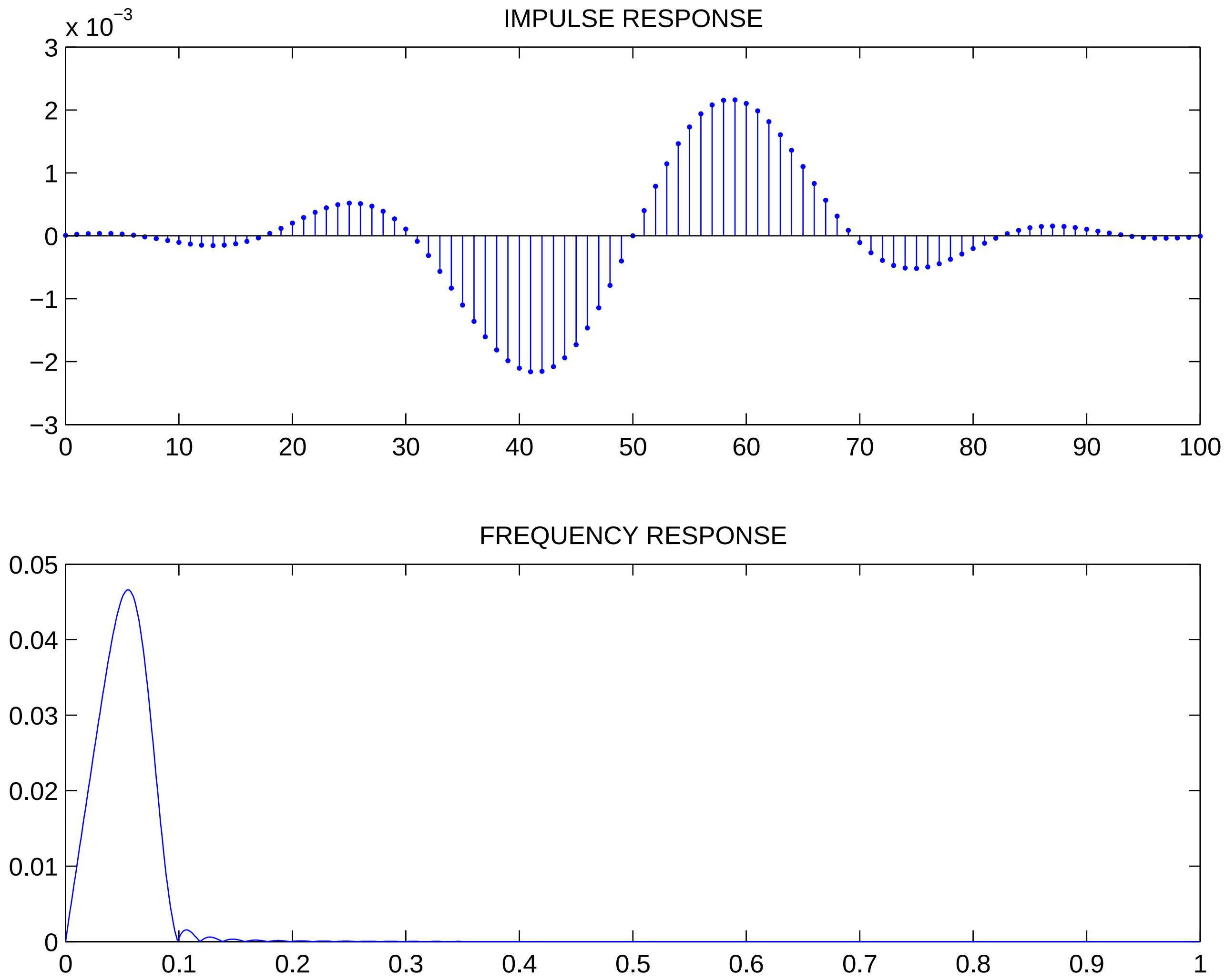
Example $K = 2$


IMPULSE RESPONSE
x 10^-3
1
0.5
0
−0.5
−1
0
10
20
30
40
50
60
70
80
90
100
FREQUENCY RESPONSE
0.025
0.02
0.015
0.01
0.005
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

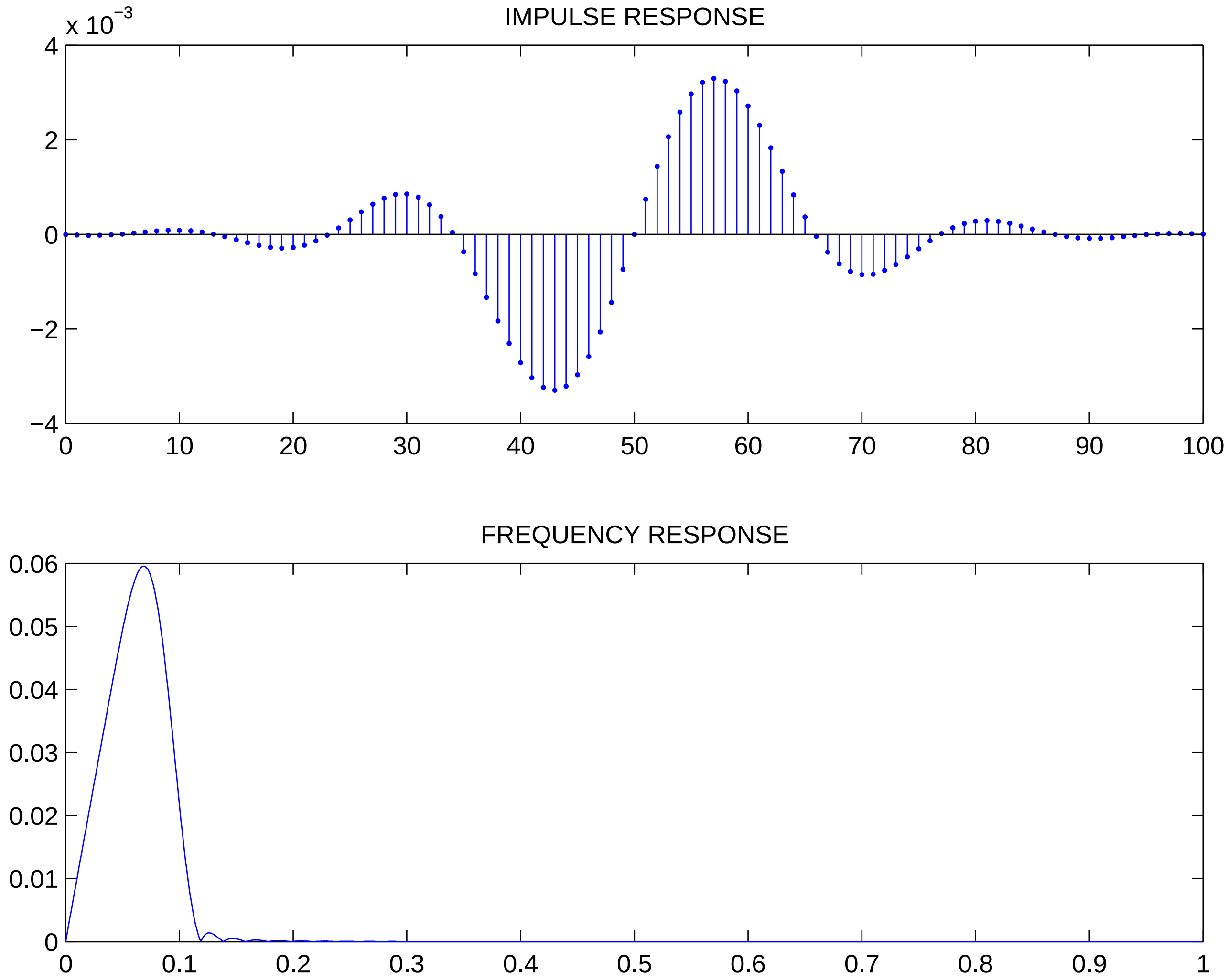
Example $K = 3$


IMPULSE RESPONSE
x 10^-3
1.5
1
0.5
0
−0.5
−1
−1.5
0
10
20
30
40
50
60
70
80
90
100
FREQUENCY RESPONSE
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

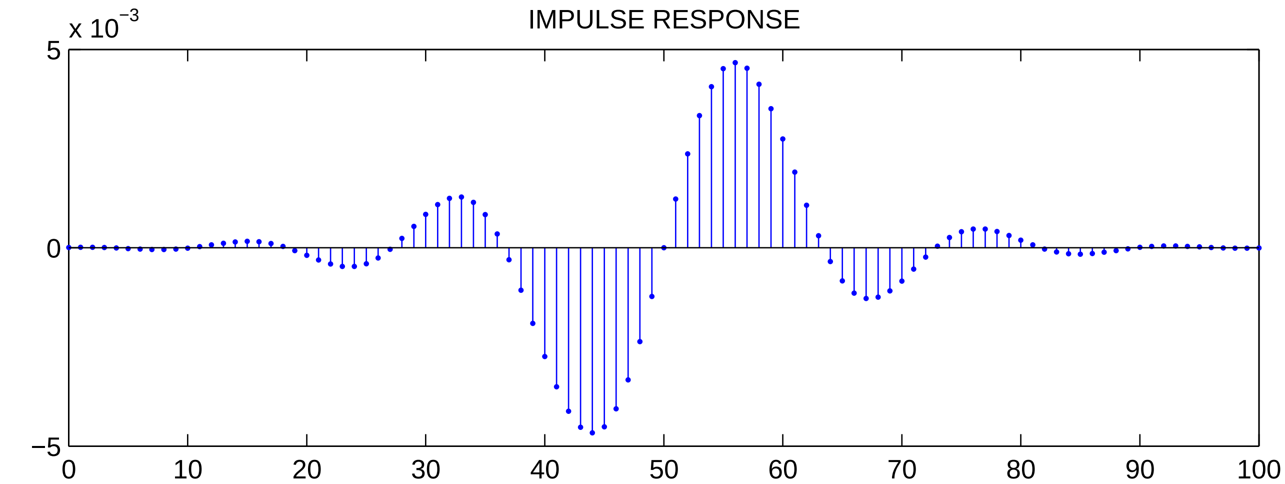
Example $K = 4$

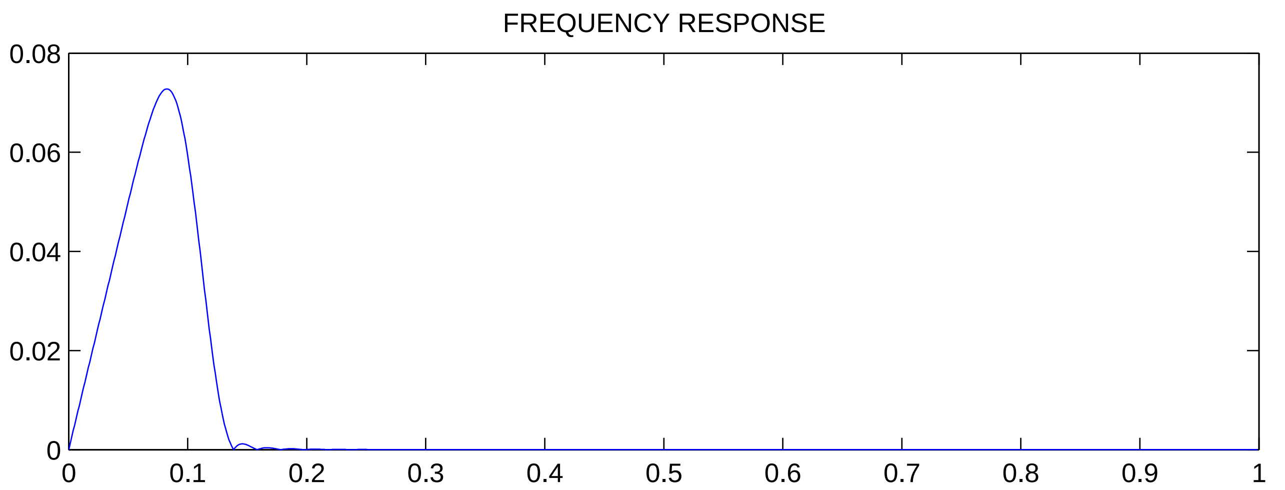

IMPULSE RESPONSE
x 10^-3
3
2
1
0
−1
−2
−3
0
10
20
30
40
50
60
70
80
90
100
FREQUENCY RESPONSE
0.05
0.04
0.03
0.02
0.01
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Example $K = 5$

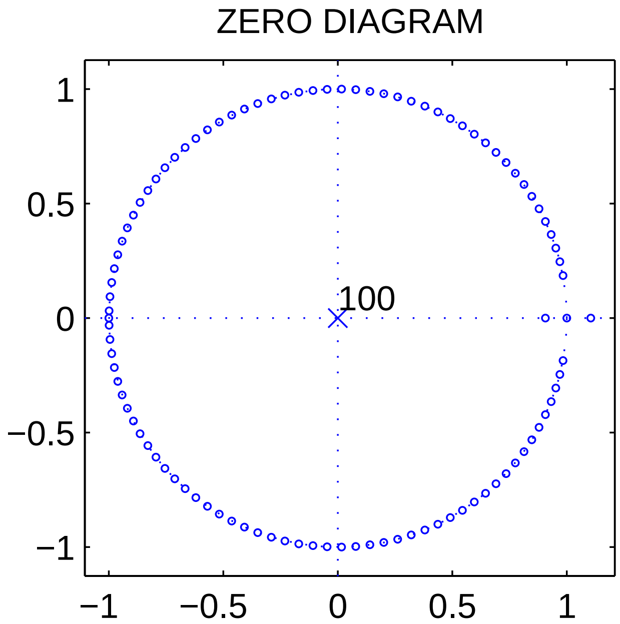

IMPULSE RESPONSE
x 10^−3
4
2
0
−2
−4
0
10
20
30
40
50
60
70
80
90
100
FREQUENCY RESPONSE
0.06
0.05
0.04
0.03
0.02
0.01
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

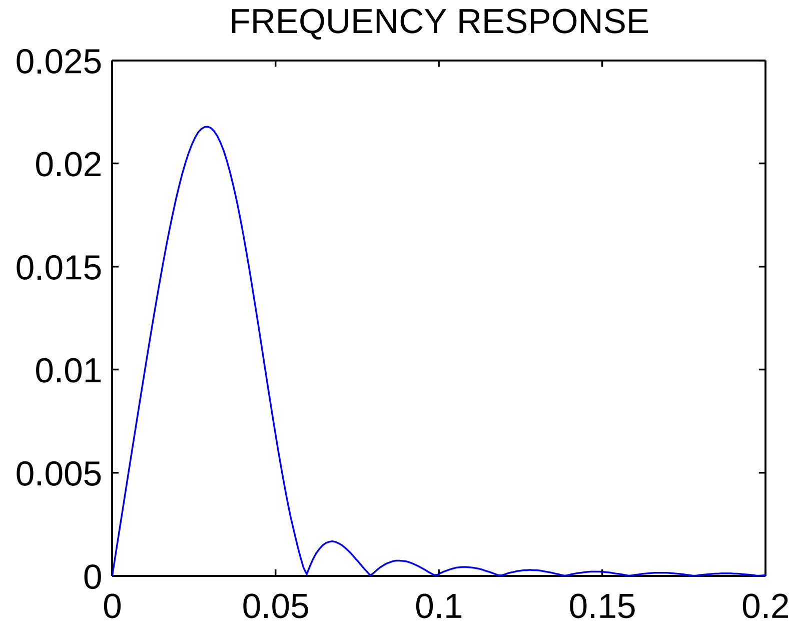
Example $K = 6$


IMPULSE RESPONSE
x 10^-3
5
0
−5
0
10
20
30
40
50
60
70
80
90
100


FREQUENCY RESPONSE
0.08
0.06
0.04
0.02
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Example $K = 2$

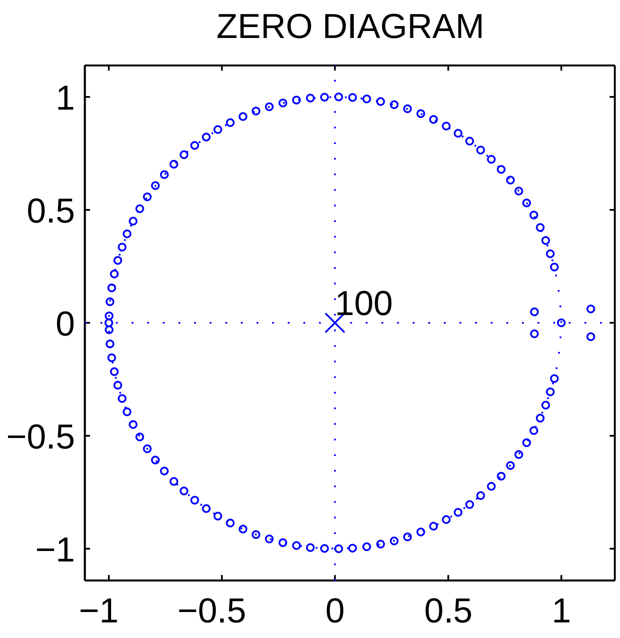


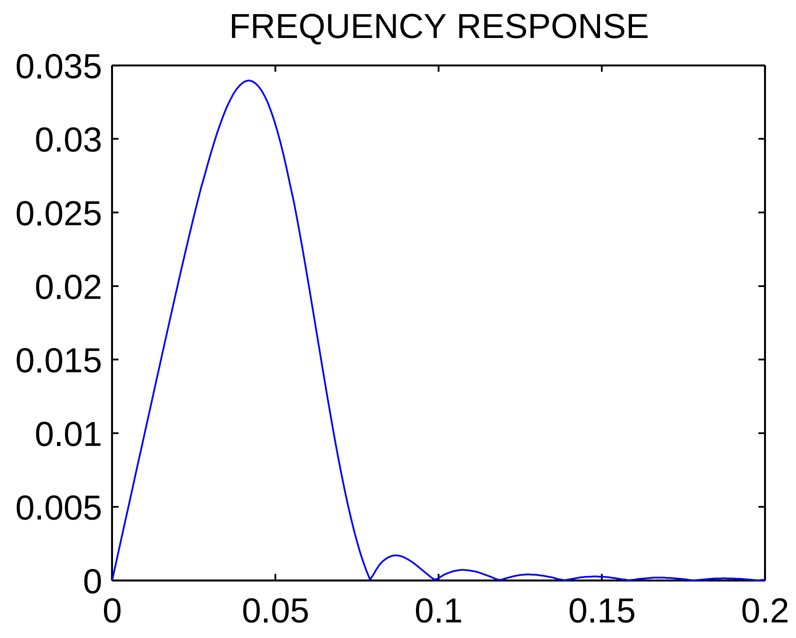


Lowpass differentiator $(K = 2,\ N = 101)$.

There are $2 = 2(K - 1)$ zeros contributing to the shape of the passband.

Example $K = 3$

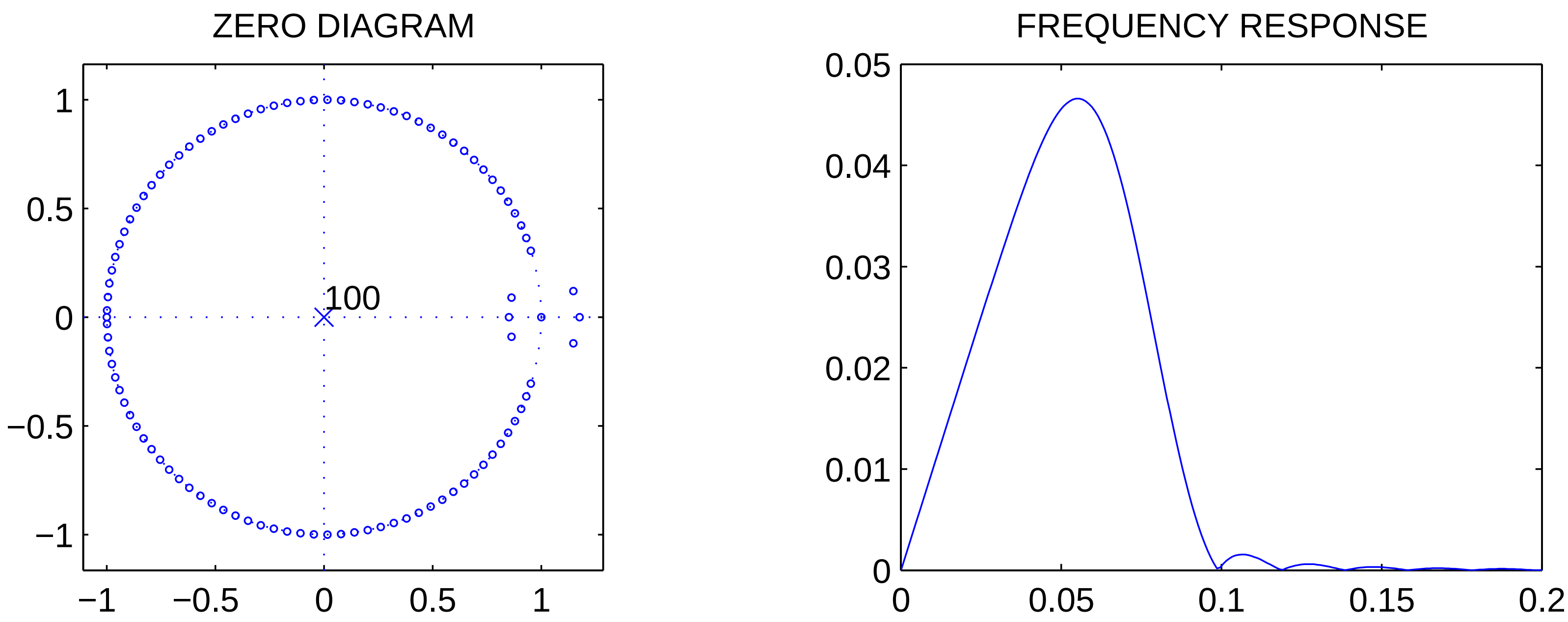




Lowpass differentiator ($K = 3$, $N = 101$).

There are $4 = 2(K-1)$ zeros contributing to the shape of the passband.

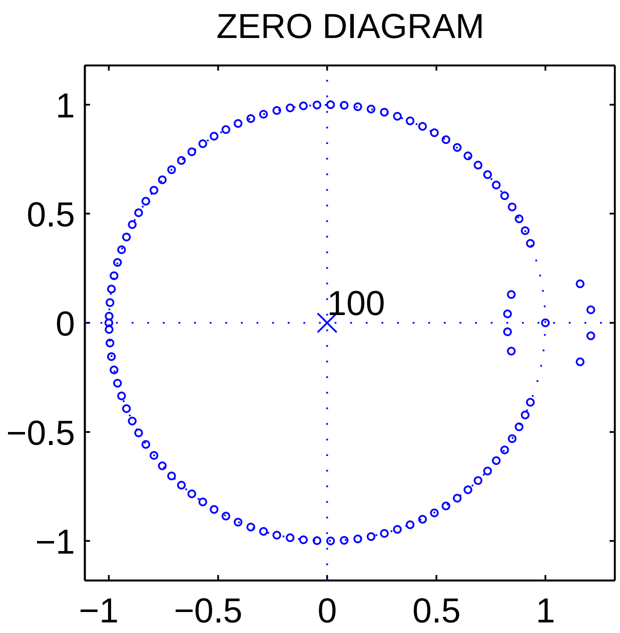
Example $K = 4$

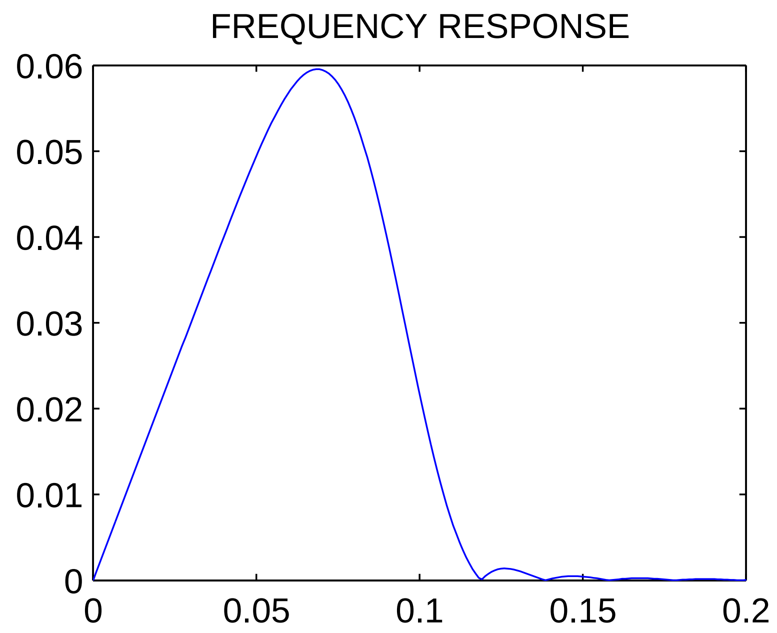


Lowpass differentiator ($K = 4,\ N = 101$).

There are $6 = 2(K-1)$ zeros contributing to the shape of the passband.

Example $K = 5$

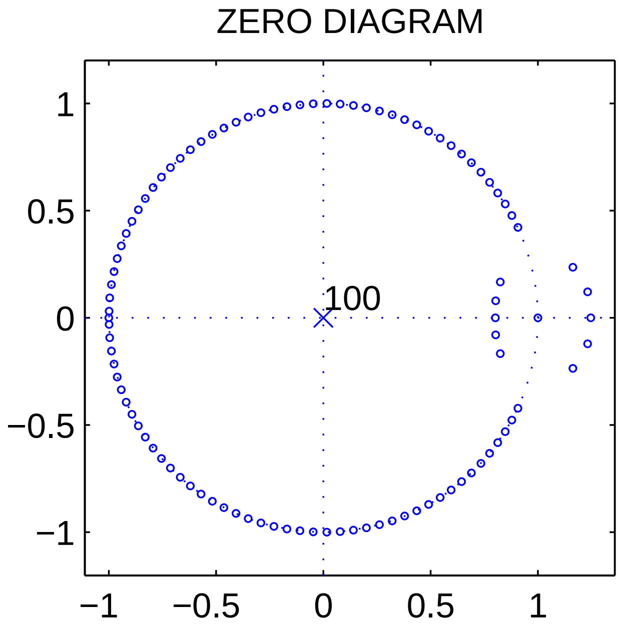


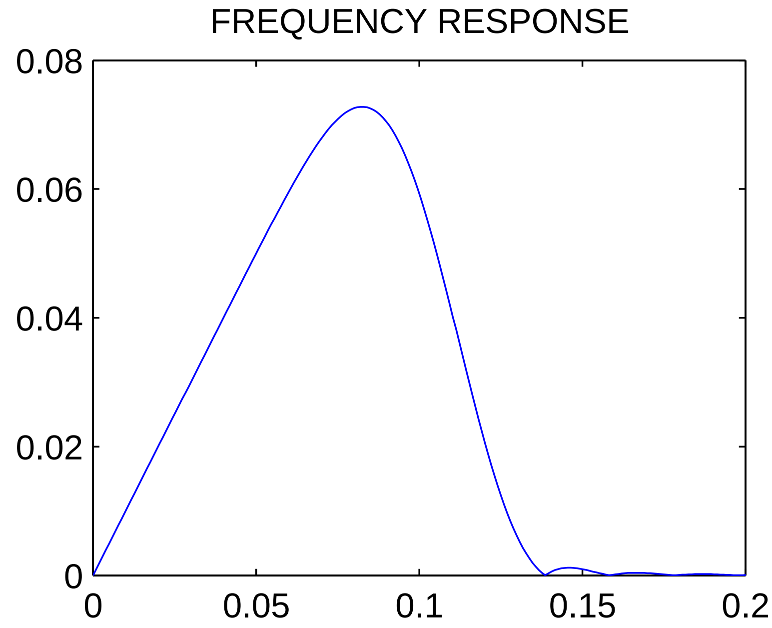


Lowpass differentiator ($K = 5,\ N = 101$).

There are $8 = 2(K-1)$ zeros contributing to the shape of the passband.

Example $K = 6$





Lowpass differentiator ($K = 6$, $N = 101$).

There are $10 = 2(K - 1)$ zeros contributing to the shape of the passband.

Summary

1. Simple Design Algorithm
2. Efficient Implementation — Frequency Sampling
3. Still needed: Design rules — How to choose N and K so that specifications are satisfied.
4. Closed form formulas for $a(k, K)$?
5. Smaller stopband ripple can be achieved by appropriate modification.