Narrowband Lowpass Digital Differentiator Design

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Abstract

This paper describes a simple formulation for the non-iterative design of narrow-band FIR linear-phase lowpass digital differentiators. The frequency response of the filters are flat around dc and have equally spaced nulls in the stopband. The design problem is formulated so as to avoid the complexity or ill-conditioning of standard methods for the design of similar filters when those methods are used to design narrow-band filters with long impulse responses.

1 Introduction

To avoid the undesirable amplification of noise in digital differentiation, lowpass differentiators can be used in place of full-band ones. Lowpass differentiators are used, for example, in airborne laser bathymetry [15]. The design of full-band digital differentiators according to the maximally-flat criterion has been described by several authors [1, 5, 6, 8]. The design of lowpass differentiators can be performed with least squares, the Remez algorithm [10] and other methods [7, 11, 13, 9]. The Kaiser window [3] is used to design lowpass differentiators in [15]. The design of maximally-flat lowpass differentiators is described in [12]. The design of FIR filters with flat passband and equiripple stopbands is described in [4, 9, 13, 14].

This paper describes a simple formulation for the non-iterative design of narrow-band FIR linear-phase lowpass digital differentiators. The filters are flat around dc and have equally spaced nulls in the stopband. The impulse response can be written as a sum of sines. The design problem is formulated so as to avoid the complexity or ill-conditioning of standard methods for the design of similar filters when those methods are used to design narrow-band filters with long impulse responses.

The frequency response of the ideal lowpass digital differentiator is

$$H_{LP}(e^{j\omega}) = \begin{cases} j\,\omega & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| < \pi \end{cases} \qquad (1)$$

It is a narrow-band filter if ω_c is much smaller than π. Note that a narrow-band filter should have a long impulse response if its frequency response is to closely approximate the ideal response. For long impulse response it is desirable to have simple design algorithms so that ill-conditioning and computational complexity is minimized. The window method for FIR filter design is a natural choice in this case. The design method described here gives an alternative approach.

Figure 2 illustrates the result of filtering an EOG signal with both a full-band differentiator and a narrow-band lowpass differentiator (with impulse response length $N = 151$ and $K = 3$ below). As shown in the figure, differentiation with the full-band differentiator yields an extremely noisy signal, while lowpass differentiation gives a more useful result.

2 Derivation

To design a narrow-band digital filter we can begin with an analog filter, the frequency response of which is denoted $A(f)$. While the frequency response $A(f)$ proposed below is not the frequency response of any realizable analog filter, it can be converted into a realizable digital filter. Provided, the filter is narrow-band, the digital-to-analog conversion used below will preserve the shape of the frequency response.

The digital filter design procedure we propose begins with an analog frequency response having the following form:

$$A(f) = \sum_{k=1}^{K} a(k,K)\ (\mathrm{sinc}(f-k) - \mathrm{sinc}(f+k))$$

where the sinc function is given by

$$\mathrm{sinc}(f) := \frac{\sin(\pi f)}{\pi f}.$$

The function $\mathrm{sinc}(f)$ is symmetric ($\mathrm{sinc}(-f) = \mathrm{sinc}(f)$) and equal to zero for $f = \pm 1, \pm 2, \pm 3, \ldots$;

therefore if we define

$$s_k(f) := \operatorname{sinc}(f-k) - \operatorname{sinc}(f+k)$$

then we have

1. $s_k(f)$ is antisymmetric, $\operatorname{sinc}(-f) = -\operatorname{sinc}(f)$.
2. $s_k(f) = 0$ for $f \in \mathbb{Z}/\{\pm k\}$. ($s_k(f) = 0$ whenever f is an integer different from $\pm k$.)

Therefore, the frequency response $A(f)$ has the following properties:

1. $A(f)$ is antisymmetric, $A(-f) = -A(f)$.
2. $A(f) = 0$ for $f = 0$, and for $f = \pm(K+1), \pm(K+2), \pm(K+3), \dots$.

The frequency response $A(f)$ is zero at $f = 0$ in agreement with a differentiator. The stopband of $A(f)$ is neither equiripple nor maximally flat. The first null in the stopband depends on K. The exact behavior of $A(f)$ depends on the coefficients $a(k,K)$, however, the uniformly spaced nulls in the stopband ensures that the attenuation increases with frequency.

The coefficients $a(k,K)$ are to be determined so that the frequency response $A(f)$ approximates f near $f = 0$: $A(f) \approx f$. The design problem for a lowpass differentiator that is flat at dc is given as follows: Given K, find $a(k,K)$ for $1 \le k \le K$ such that the derivatives of $A(f)$ at $f = 0$ match the derivatives of the ideal differentiator $\operatorname{IdealDiff}(f) := f$ at the point $f = 0$:

$$A^{(1)}(0) = 1 \tag{2}$$

$$A^{(i)}(0) = 0, \qquad i = 3, 5, \dots, 2K-1. \tag{3}$$

The even derivatives are automatically zero because $A(f)$ is an odd function, $A(-f) = -A(f)$. This is a linear system of equations with an equal number of equations and variables.

For example, when $K = 1$, we have $a(1,1) = \frac{1}{2}$. When $K = 2$, we have

$$a(1,2) = -\frac{1}{6} + \frac{1}{9}\pi^2 \tag{4}$$

$$a(2,2) = -\frac{4}{3} + \frac{2}{9}\pi^2 \tag{5}$$

When $K = 3$, we have

$$a(1,3) = \frac{1}{48} - \frac{13}{288}\pi^2 + \frac{7}{480}\pi^4 \tag{6}$$

$$a(2,3) = \frac{16}{15} - \frac{16}{9}\pi^2 + \frac{14}{75}\pi^4 \tag{7}$$

$$a(3,3) = \frac{243}{80} - \frac{81}{32}\pi^2 + \frac{189}{800}\pi^4 \tag{8}$$

When $K = 4$, we have

$$a(1,4) = -\frac{1}{720} + \frac{29}{4320}\pi^2 - \frac{427}{64800}\pi^4 + \frac{31}{18900}\pi^6$$
$$a(2,4) = -\frac{16}{45} + \frac{208}{135}\pi^2 - \frac{2366}{2025}\pi^4 + \frac{496}{4725}\pi^6$$
$$a(3,4) = -\frac{2187}{560} + \frac{2187}{160}\pi^2 - \frac{5103}{800}\pi^4 + \frac{2511}{4900}\pi^6$$
$$a(4,4) = -\frac{2048}{315} + \frac{2048}{135}\pi^2 - \frac{12544}{2025}\pi^4 + \frac{15872}{33075}\pi^6$$

Other values $a(k,K)$ can be easily computed. Numerical values of the first coefficients (for K up to 8) are given in Table 1.

3 Conversion to Digital Filter

To convert the analog frequency response $A(f)$ into a digital frequency response $D(f)$, we can use the digital sinc function in place of the usual sinc function. The digital sinc function $\operatorname{dsinc}(f, N)$ can be written as

$$\operatorname{dsinc}(f, N) := \frac{\sin(N\pi f)}{\sin(\pi f)}. \tag{9}$$

The digital sinc function defined in (9) is periodic in f with period 2:

$$\operatorname{dsinc}(f+2) = \operatorname{dsinc}(f).$$

The definition (9) is not the standard form of the digital sinc function — the only difference is a scaling. However, defining dsinc in this way, we have the following approximation:

$$\operatorname{sinc}(f) \approx \frac{1}{N}\operatorname{dsinc}\left(\frac{f}{N}, N\right) \quad \text{for} \quad |f| < 0.5\,N.$$

This approximation is illustrated in Figure 3 for $N = 30$ and $N = 100$. As illustrated in the figure, the approximation is more accurate for greater values of N. The design of digital differentiators described in this paper is intended for long impulses responses. In this case, N is large and the approximation is valid.

4 Digital Lowpass Differentiators

Consider the function $D(f)$, based on the digital sinc function:

$$D(f) = \frac{1}{N}\sum_{k=1}^{K} a(k,K)\left[\operatorname{dsinc}\left(\frac{f-k}{N}, N\right) - \operatorname{dsinc}\left(\frac{f+k}{N}, N\right)\right] \tag{10}$$

For $N > K$, the function $D(f)$ approximates the function $A(f)$ in the range $|f| < N/2$. To illustrate this approximation for $K = 3$, $N = 30$, Figure 1 shows the difference $A(f) - D(f)$ on the range $|f| < N/2$. The figure shows that $D(f) \approx A(f)$ for $|f| < N/2$. The function $D(\frac{N}{2\pi}\omega)$ is then a 2π periodic function of ω and can therefore be used as the frequency response $H(e^{j\omega})$ of a digital filter. Then $H(e^{j\omega})$ will be approximately maximally flat at $\omega = 0$. The impulse response $h(n)$ is given by the inverse discrete-time Fourier transform of $H(e^{j\omega})$,

$$h(n) = \mathrm{IDTFT}\left\{ e^{-j(\frac{N-1}{2})\omega} \cdot D\left(\frac{N}{2\pi}\,\omega\right) \right\}$$

where the phase term is included to make $h(n)$ causal. Then $h(n)$ is a linear-phase FIR impulse response of length N:

$$h(n) = \frac{1}{N^2} \sum_{k=1}^{K} a(k,K)\, \sin\left(\frac{2\pi k}{N}\left(n - \frac{(N-1)}{2}\right)\right)$$

for $0 \le n \le N-1$. This gives a very simple way to design narrow-band digital differentiators. Once a table of values $a(k,K)$ is computed, it can be used regardless of the length N of the impulse response $h(n)$. The first values (for K up to 8) are given in Table 1.

The width of the passband is controlled by the parameter K, the number of flatness constraints at dc. Note that $2\,(K-1)$ is the number of zeros of $H(z)$ that lie away from the unit circle, as illustrated in the following examples.

Figures 4 and 5 illustrate two digital lowpass differentiators designed according to the method described above. The figures illustrate the impulse response $h(n)$, the frequence response $|H(e^{j\omega})|$, and the zero diagram. The fourth panel in each figure illustrates the magnification of the passband of the differentiators.

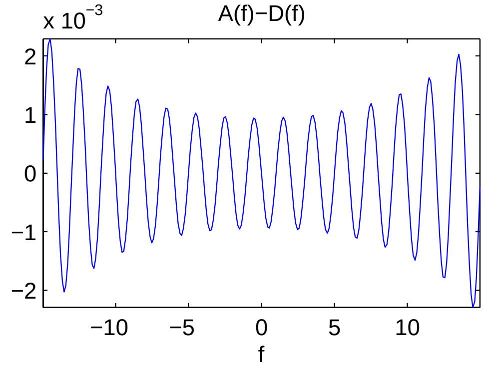

Figure 1: The function $A(f) - D(f)$; $K = 3$, $N = 30$. For large values of N, $D(f)$ is a good approximation to $A(f)$ for $|f| < 0.5N$.

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K	$a(1,K)$	$a(2,K)$	$a(3,K)$	$a(4,K)$	$a(5,K)$	$a(6,K)$	$a(7,K)$	$a(8,K)$
1	0.5000							
2	0.9300	0.8599						
3	0.9959	1.7037	1.0680					
4	0.9999	1.9591	2.3145	1.1673				
5	1.0000	1.9966	2.8547	2.7662	1.1913			
6	1.0000	1.9998	2.9798	3.6548	3.0723	1.1647		
7	1.0000	2.0000	2.9980	3.9336	4.3401	3.2516	1.1058	
8	1.0000	2.0000	2.9999	3.9907	4.8393	4.8996	3.3245	1.0277

Table 1: $a(k,K)$ values for $1 \leq K \leq 8$. The Kth row of the table gives the coefficients $a(k,K)$ required to compute $A(f)$, and the impulse response $h(n)$.

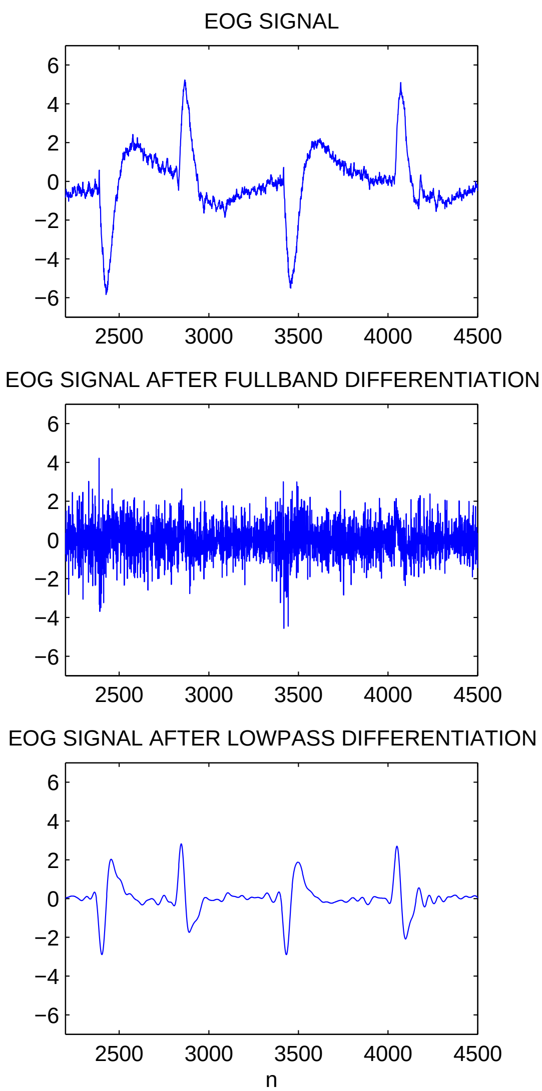

Figure 2: An EOG signal filtered using full-band and lowpass differentiators. For noisy signals, (narrow-band) lowpass differentiators yield more useful results than full-band differentiators.

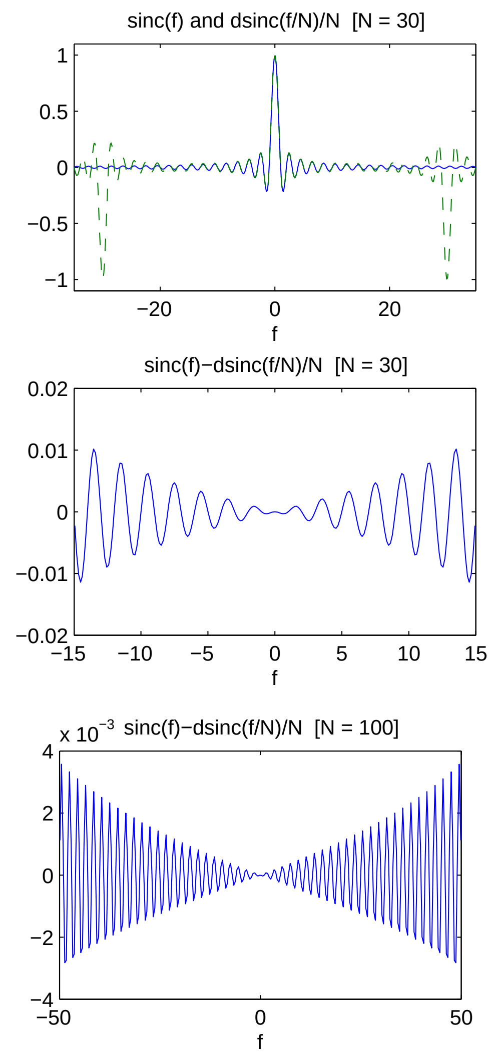

Figure 3: Comparison of sinc(f) and dsinc$(f/N)/N$ for $N = 30$ and $N = 100$. For large values of N, sinc(f) is well approximated by dsinc$(f/N)/N$ for $|f| < 0.5N$.

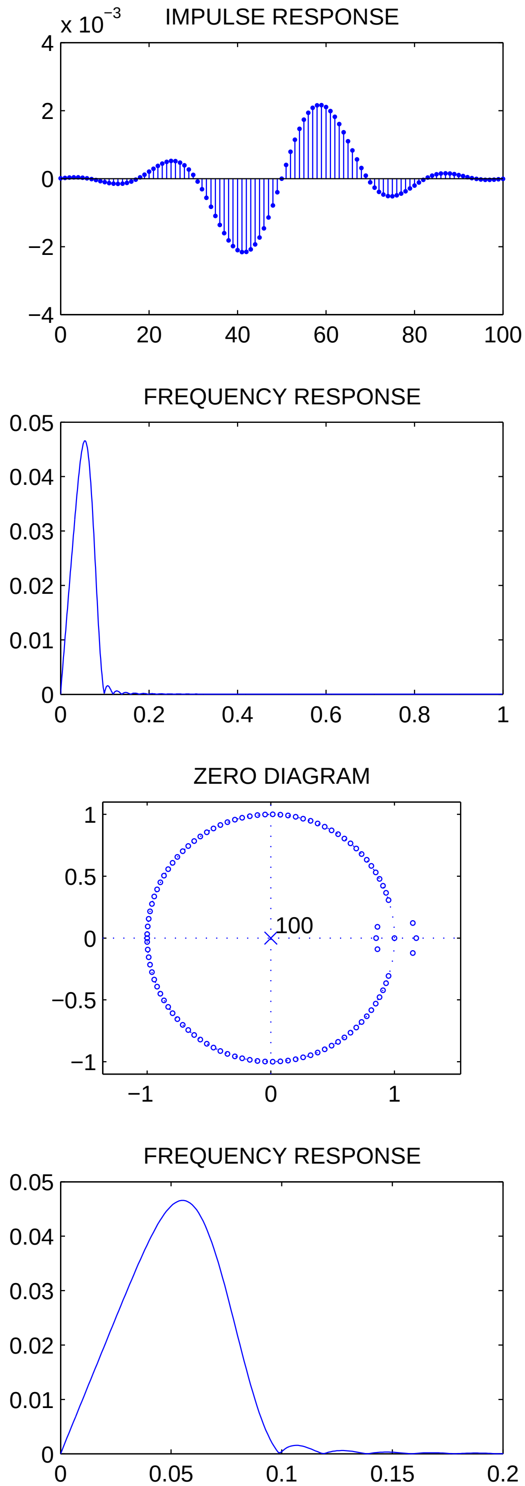

Figure 4: Lowpass differentiator ($K = 4$, $N = 101$). The length of the impulse response is $N = 101$. There are $6 = 2(K-1)$ zeros contributing to the shape of the passband.

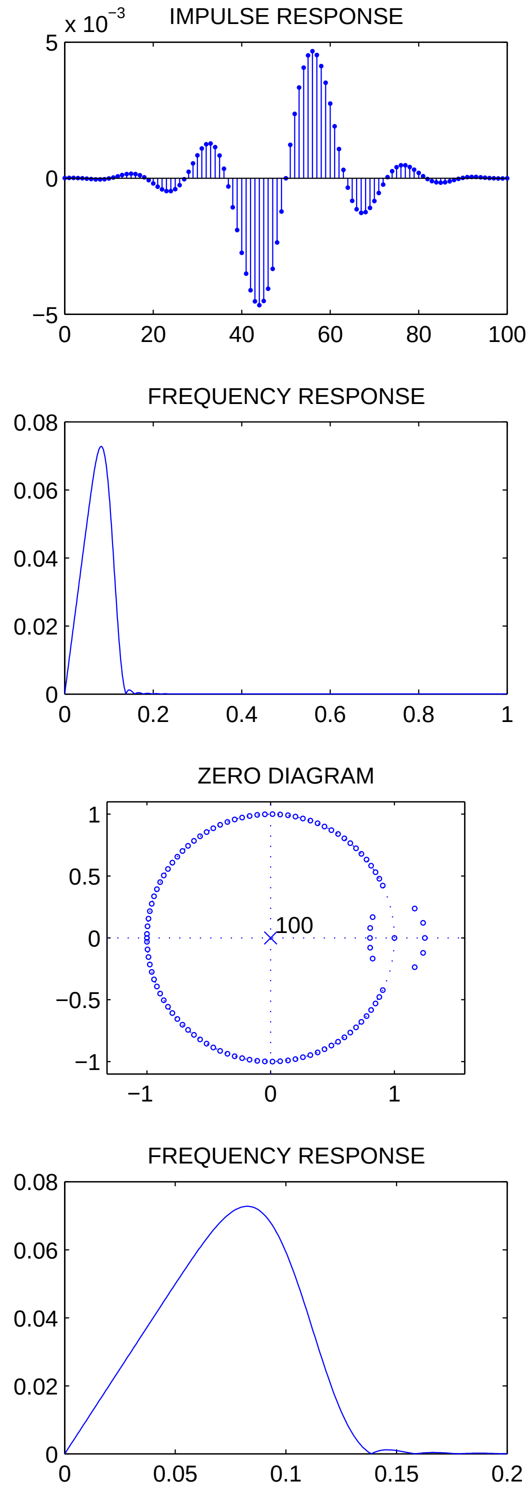

Figure 5: Lowpass differentiator ($K = 6$, $N = 101$). The length of the impulse response is $N = 101$. There are $10 = 2(K-1)$ zeros contributing to the shape of the passband. The passband is wider than that shown in Figure 4.