Maximally Flat Low-Pass Digital Differentiators

Ivan W. Selesnick

***Abstract*—This paper describes the design of type III and type IV linear-phase finite-impulse response (FIR) low-pass digital differentiators according to the maximally flat criterion. We introduce a two-term recursive formula that enables the simple stable computation of the impulse response coefficients. The same recursive formula is valid for both Type III and Type IV solutions.**

***Index Terms*—Finite-impulse response (FIR) digital filters, linear-phase filters, low-pass differentiators, low-pass filters.**

I. Introduction

This paper describes the design of linear-phase finite-impulse response (FIR) low-pass digital differentiators according to the maximally flat criterion. We introduce a two-term recursive formula that enables the simple stable computation of the impulse response coefficients. The same recursive formula is valid for both Type III and Type IV solutions.

To avoid the undesirable amplification of noise in digital differentiation, low-pass differentiators can be used in place of full-band ones. They find application, for example, in bar code readers [16]. Maximally flat digital full-band differentiators have been described by several authors [3], [7]–[9], [14] where the approximation is accurate at a single point, usually $\omega = 0, \pi/2$, or π. When the input signal lies in a known frequency band centered around the frequency ω_o, it is desirable to use a differentiator which is maximally flat at ω_o rather than at $\omega = 0$. Solutions of that type have been introduced in [2], [10]–[12]. The design of low-pass differentiators can be performed with least squares, the Remez algorithm [19] and other methods [13], [24], [25], [17]; however, the design of maximally flat low-pass differentiators has not been previously addressed.

The frequency response of the ideal (full-band) digital differentiator is

$$H_{\mathrm{FB}}(e^{j\omega}) = j\omega, \quad |\omega| < \pi. \tag{1}$$

The maximally flat FIR approximation to the ideal differentiator satisfies the derivative constraints

$$|H(e^{j\omega})| = 0, \quad \omega = 0 \tag{2}$$

$$\frac{d}{d\omega}|H(e^{j\omega})| = 1, \quad \omega = 0 \tag{3}$$

$$\frac{d^k}{d\omega^k}|H(e^{j\omega})| = 0, \quad \omega = 0, \quad 2 \le k \le B-1 \tag{4}$$

where B is the number of free parameters. Notice that the approximation is performed at a single frequency point $\omega = 0$. Kumar and Dutta Roy described how the solution can be obtained from the maximally flat low-pass FIR filter [9].

The frequency response of the ideal low-pass digital differentiator is

$$H_{\mathrm{LP}}(e^{j\omega}) = \begin{cases} j\omega, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases} \tag{5}$$

where ω_c is the cutoff frequency. This paper describes the maximally flat linear-phase FIR approximation, which satisfies the constraints

$$|H(e^{j\omega})| = 0, \quad \omega = 0 \tag{6}$$

$$\frac{d}{d\omega}|H(e^{j\omega})| = 1, \quad \omega = 0 \tag{7}$$

$$\frac{d^k}{d\omega^k}|H(e^{j\omega})| = 0, \quad \omega = 0, \quad 2 \le k \le 2L \tag{8}$$

$$\frac{d^k}{d\omega^k}|H(e^{j\omega})| = 0, \quad \omega = \pi, \quad 0 \le k \le 2M. \tag{9}$$

In this case, it is not a single point approximation problem and the solution can not be obtained directly from the maximally flat low-pass filter. Type III and Type IV linear-phase FIR solutions to this problem are derived in Sections 2 and 3 respectively. As in [14], the method used in this paper is based on power series expansion. To compute the impulse response coefficients in a simple way, Section IV gives a two-term recursive formula which is valid for both Type III and IV solutions. This recursive formula was obtained using the algorithms of [15] for automatic hyper-geometric-type sum simplification. A Matlab program to compute the maximally flat differentiator based on the two-term recurrence is available from the author. The derivation of the maximally flat low-pass differentiator below uses the transformation of variables used, for example, by Herrmann in the derivation of maximally flat low-pass Type I FIR filters [6]. Similar to [14], the expression obtained in (58) below for the transfer function can be used as the basis of a structure for an efficient implementation of the filter.

II. Type IV Solution

The derivation of the solution will depend on a transformation that maps polynomials on the real interval [0, 1] to polynomials on the upper half of the unit circle. Let $P(x)$ be a polynomial

$$P(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n. \tag{10}$$

If we define

$$H(z) = P\left(\frac{-z + 2 - z^{-1}}{4}\right) \tag{11}$$

then $H(z)$ will be a Type I transfer function and its frequency response is given by

$$H(e^{j\omega}) = P\left(\frac{1}{2} - \frac{1}{2}\cos(\omega)\right). \tag{12}$$

Now, note that a type IV transfer function $H_4(z)$ can always be written as

$$H_4(z) = \frac{1}{2}(1 - z^{-1})H(z). \tag{13}$$

(This is because a Type IV transfer function always has a zero at $z = 1$. See [13] and [18], for example, for a description of the four types of FIR linear-phase digital filter.) The frequency response of $H_4(z)$ is given by

$$H_4(e^{j\omega}) = \frac{1}{2}(1 - e^{-j\omega})H(e^{j\omega}) \tag{14}$$

$$= j \cdot e^{-j\frac{\omega}{2}} \sin\left(\frac{\omega}{2}\right) H(e^{j\omega}) \tag{15}$$

$$= j \cdot e^{-j\frac{\omega}{2}} \sqrt{\frac{1}{2} - \frac{1}{2}\cos(\omega)} \cdot P\left(\frac{1}{2} - \frac{1}{2}\cos(\omega)\right) \tag{16}$$

$$= j \cdot e^{-j\frac{\omega}{2}} P_4\left(\frac{1}{2} - \frac{1}{2}\cos(\omega)\right) \tag{17}$$

Manuscript received August 30, 2001; revised March 28, 2002. This work was supported in part by the National Science Foundation (NSF) under CAREER Grant CCR-987452. This paper was recommended by Associate Editor A. Skodras.

The author is with the Electrical Engineering Department, Polytechnic University, Brooklyn, NY 11201 USA (e-mail: selesi@taco.poly.edu).

Publisher Item Identifier S 1057-7130(02)05967-0.

where

$$P_4(x) := \sqrt{x}P(x). \tag{18}$$

Note that even though the term $e^{-j(\omega/2)}$ appears in the expression for $H_4(e^{j\omega})$ no fractional delay element is needed for the implementation of this digital filter. ($H(z)$ is a Type I transfer function which requires no fractional delays, and (13) does not introduce any fractional delay.)

Our goal is to design $P(x)$ so that $H_4(e^{j\omega})$ approximates the ideal low-pass differentiator in the maximally flat sense. That means it is desired that

$$P_4\left(\frac{1}{2} - \frac{1}{2}\cos(\omega)\right) \approx \begin{cases} \omega, & \text{for } \omega \approx 0 \\ 0, & \text{for } \omega \approx \pi. \end{cases} \tag{19}$$

Equivalently,

$$P_4(x) \approx \begin{cases} \arccos(1-2x), & \text{for } x \approx 0 \\ 0, & \text{for } x \approx 1 \end{cases} \tag{20}$$

or in terms of $P(x)$,

$$P(x) \approx \begin{cases} \dfrac{\arccos(1-2x)}{\sqrt{x}}, & \text{for } x \approx 0 \\ 0, & \text{for } x \approx 1. \end{cases} \tag{21}$$

These approximation condition can be written as

$$P(x) = A(x) \cdot (1-x)^M \tag{22}$$

and

$$\frac{\arccos(1-2x)}{\sqrt{x}} - P(x) = B(x) \cdot x^{L+1} \tag{23}$$

where M, and $L+1$ are the degrees of tangency at $x=1$ and $x=0$. The parameters M and L are the parameters that define the solution. $A(x)$ and $B(x)$ are power series in x. Adding (22) and (23), we get

$$\frac{\arccos(1-2x)}{\sqrt{x}} = B(x) \cdot x^{L+1} + A(x) \cdot (1-x)^M. \tag{24}$$

Solving for $A(x)$ we get

$$A(x) = \frac{1}{(1-x)^M} \cdot \left(\frac{\arccos(1-2x)}{\sqrt{x}} - B(x) \cdot x^{L+1}\right). \tag{25}$$

Let the Taylor series of the first term be given by

$$\frac{1}{(1-x)^M} \cdot \frac{\arccos(1-2x)}{\sqrt{x}} = \sum_{n=0}^{\infty} c(n)x^n. \tag{26}$$

Note that the polynomial $A(x)$ of minimal degree which satisfies (25) will be of degree L. (Suppose the degree of $A(x)$ were less than L, then it would be impossible to choose a power series $B(x)$ so that the right-hand side of (25) has the same degree as the left-hand side.) With $A(x)$ being minimal degree, we have

$$A(x) = \sum_{n=0}^{L} c(n)x^n \tag{27}$$

$$P(x) = (1-x)^M \sum_{n=0}^{L} c(n)x^n \tag{28}$$

and

$$H_4(z) = \frac{1}{2}(1-z^{-1})H(z) \tag{29}$$

$$= \frac{1}{2}(1-z^{-1})P\left(\frac{-z+2-z^{-1}}{4}\right) \tag{30}$$

$$= \frac{1}{2}(1-z^{-1})\left(\frac{z+2+z^{-1}}{4}\right)^M \times \sum_{n=0}^{L} c(n)\left(\frac{-z+2-z^{-1}}{4}\right)^n. \tag{31}$$

Note that (1) the degree of $P(x)$ is $M+L$, (2) the degree of $H_4(z)$ is $2M+2L+1$. That makes $h_4(n)$ an impulse response of length $2M+2L+2$. We need only find the Taylor series coefficients $c(n)$. To this end, we define

$$A_1(x) := \frac{1}{(1-x)^M} = \sum_{n=0}^{\infty} c_1(n)x^n \tag{32}$$

where

$$c_1(n) = \binom{M+n-1}{n} \tag{33}$$

and

$$A_2(x) := \frac{\arccos(1-2x)}{\sqrt{x}} = \sum_{n=0}^{\infty} c_2(n)x^n \tag{34}$$

where

$$c_2(n) = \frac{2}{2n+1} \cdot \frac{1}{n!} \cdot \left(\frac{1}{2}\right)_n \tag{35}$$

which is derived in the Appendix. Then $A(x) = A_1(x)A_2(x)$ and

$$c(n) = \sum_{k=0}^{n} c_1(k)c_2(n-k). \tag{36}$$

In Section IV, we give a recursive formula for computing the coefficients $c(n)$.

III. Type III Solution

Note that a type III transfer function $H_3(z)$ can always be written as

$$H_3(z) = \frac{1}{2}(1-z^{-1})\frac{1}{2}(1+z^{-1})H(z) \tag{37}$$

where $H(z)$ is a Type I transfer function. (This is because a Type III transfer function always has a zero at $z=1$ and $z=-1$.) The frequency response of $H_3(z)$ is given by

$$H_3(e^{j\omega}) = j \cdot e^{-j\frac{\omega}{2}} \sin\left(\frac{\omega}{2}\right) e^{-j\frac{\omega}{2}} \cos\left(\frac{\omega}{2}\right) H(e^{j\omega}) \tag{38}$$

$$= j \cdot e^{-j\omega}\sqrt{\frac{1}{2} - \frac{1}{2}\cos(\omega)} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\cos(\omega)} \cdot P\left(\frac{1}{2} - \frac{1}{2}\cos(\omega)\right) \tag{39}$$

$$= j \cdot e^{-j\frac{\omega}{2}} P_3\left(\frac{1}{2} - \frac{1}{2}\cos(\omega)\right) \tag{40}$$

where

$$P_3(x) := \sqrt{x}\sqrt{1-x}P(x). \tag{41}$$

To design $P(x)$ so that $H_3(e^{j\omega})$ approximates the ideal low-pass differentiator, we ask that

$$P_3\left(\frac{1}{2} - \frac{1}{2}\cos(\omega)\right) \approx \begin{cases} \omega, & \text{for } \omega \approx 0 \\ 0, & \text{for } \omega \approx \pi. \end{cases} \tag{42}$$

Equivalently,

$$P_3(x) \approx \begin{cases} \arccos(1-2x) & \text{for } x \approx 0 \\ 0 & \text{for } x \approx 1 \end{cases} \tag{43}$$

or in terms of $P(x)$,

$$P(x) \approx \begin{cases} \dfrac{\arccos(1-2x),}{\sqrt{x}\sqrt{1-x}} & \text{for } x \approx 0 \\ 0, & \text{for } x \approx 1. \end{cases} \tag{44}$$

These approximation condition can be written as

$$P(x) = A(x) \cdot (1-x)^M \tag{45}$$

and

$$\frac{\arccos(1-2x)}{\sqrt{x}\sqrt{1-x}} - P(x) = B(x) \cdot x^{L+1} \tag{46}$$

where M, and $L+1$ are the degrees of tangency at $x=1$ and $x=0$. Adding (45) and (46), we get

$$\frac{\arccos(1-2x)}{\sqrt{x}\sqrt{1-x}} = B(x) \cdot x^{L+1} + A(x) \cdot (1-x)^M. \tag{47}$$

As above, the polynomial $A(x)$ of minimal degree which satisfies (47) will be of degree L. Solving for $A(x)$ of minimal degree L we get

$$A(x) = \sum_{n=0}^{L} c(n)x^n \tag{48}$$

where $c(n)$ are the coefficients in the Taylor series of

$$\frac{\arccos(1-2x)}{\sqrt{x}(1-x)^{M+\frac{1}{2}}} = \sum_{n=0}^{\infty} c(n)x^n. \tag{49}$$

Then

$$H_3(z) = \frac{1}{2}(1-z^{-1})\frac{1}{2}(1+z^{-1})H(z) \tag{50}$$

$$= \frac{1}{4}(1-z^{-1})(1+z^{-1})P\left(\frac{-z+2-z^{-1}}{4}\right) \tag{51}$$

$$= \frac{1}{4}(1-z^{-1})(1+z^{-1})\left(\frac{z+2+z^{-1}}{4}\right)^M \times \sum_{n=0}^{L} c(n)\left(\frac{-z+2-z^{-1}}{4}\right)^n. \tag{52}$$

Note that (1) $P(x)$ is of degree $M+L$ again, and (2) $H_3(z)$ is of degree $2M+2L+2$. That makes $h_3(n)$ an impulse response of length $2M+2L+3$. To determine $c(n)$, define $A_1(x)$ as

$$A_1(x) := \frac{1}{(1-x)^{M+\frac{1}{2}}} = \sum_{n=0}^{\infty} c_1(n)x^n \tag{53}$$

where

$$c_1(n) = \binom{M+\frac{1}{2}+n-1}{n} \tag{54}$$

and define $A_2(x)$ as in (34). Then $A(x) = A_1(x)A_2(x)$ and $c(n) = \sum_{k=0}^{n} c_1(k)c_2(n-k)$. The coefficients $c_2(n)$ are the same for Type III and Type IV solutions, while $c_1(n)$ has a different form.

To evaluate the binomial coefficient for fractional values of the upper entry, we can use the Gamma function Γ

$$c_1(n) = \binom{M+\frac{1}{2}+n-1}{n} = \frac{\Gamma\left(M+\frac{1}{2}+n\right)}{\Gamma\left(M+\frac{1}{2}\right)\Gamma(n+1)}. \tag{55}$$

The Gamma function also satisfies $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, so we can write a recursive equation for evaluating $c_1(n)$

$$c_1(n) = \frac{1}{n}(M+n-1/2) \cdot c_1(n-1). \tag{56}$$

In Section IV we unify the Type III and Type IV solutions.

TABLE I
NUMERICAL VALUES OF THE WEIGHTS $c(n)$ FOR $0 \le K \le 3, 0 \le n \le 10$

	$c(n)$			
n	$K=0$	$K=1$	$K=2$	$K=3$
0	2.0000	2.0000	2.0000	2.0000
1	0.3333	1.3333	2.3333	3.3333
2	0.1500	1.0667	2.4833	4.4000
3	0.0893	0.9143	2.5726	5.3143
4	0.0608	0.8127	2.6334	6.1270
5	0.0447	0.7388	2.6781	6.8658
6	0.0347	0.6820	2.7128	7.5478
7	0.0279	0.6365	2.7408	8.1843
8	0.0231	0.5991	2.7639	8.7834
9	0.0195	0.5675	2.7834	9.3509
10	0.0168	0.5405	2.8002	9.8914

IV. COMBINED FORMULA

Let K denote the number of zeros a transfer function has at $z=-1$. A Type IV transfer function always has an even number of zeros at $z=-1$; for the solution given in Section III we have $K=2M$. On the other hand, a Type III transfer function always has an odd number of zeros at $z=-1$; for the solution given in Section II we have $K=2M+1$. In terms of K, the coefficients $c_1(n)$ in (33) and (54) are given by the same formula, namely

$$c_1(n) = \binom{K/2+n-1}{n}. \tag{57}$$

A causal maximally flat low-pass differentiator with linear phase is given by

$$H(z) = \left(\frac{1-z^{-1}}{2}\right)\left(\frac{1+z^{-1}}{2}\right)^K z^{-L} \times \sum_{n=0}^{L} c(n)\left(\frac{-z+2-z^{-1}}{4}\right)^n \tag{58}$$

with

$$c(n) = \sum_{k=0}^{n} c_1(k)c_2(n-k) \tag{59}$$

where $c_1(n)$ is given by (57) and $c_2(n)$ is given by (35). Using the algorithms given in [15] we can find the following two-term recursive formula for computing the coefficients $c(n)$, see (60) at the bottom of the page, for $n \ge 2$ with $c(0)=2$, and $c(1)=K+1/3$. The formula in (58) with the recursion for $c(n)$ in (60) generates both Type III and Type IV solutions. When K is even we obtain a Type IV transfer function, when K is odd we obtain a Type III transfer function. In either case, the length of the impulse response is $N=K+2L+2$. Table I gives numerical values for $c(n)$ for several values of K and n. Notice that $c(n)$ does not depend on L; rather L determines how many values

$$c(n) = \frac{(8n^2+4Kn-10n-K+3)c(n-1)-(2n+K-3)^2c(n-2)}{2n(2n+1)} \tag{60}$$

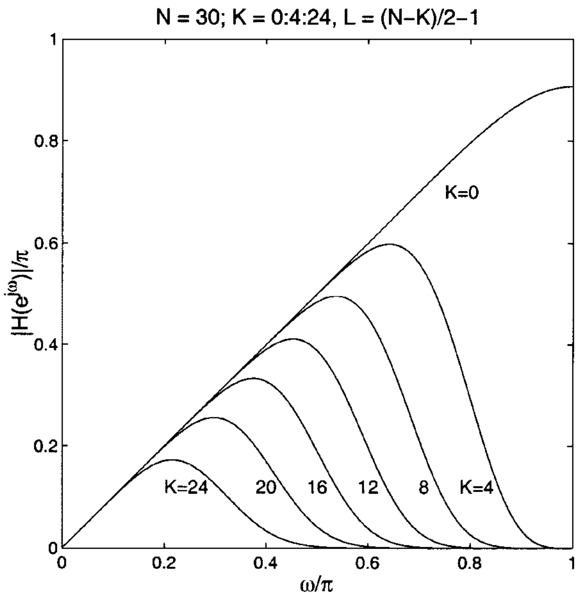

Fig. 1. Type IV maximally flat low-pass differentiators.

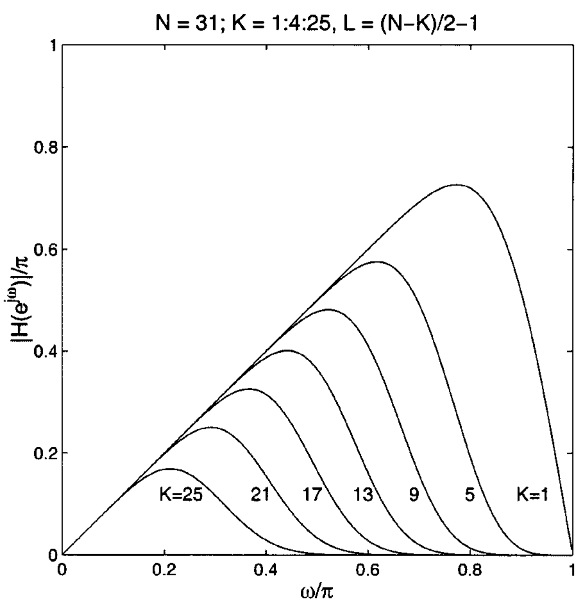

Fig. 2. Type III maximally flat low-pass differentiators.

of $c(n)$ are needed in (58). As noted in [14], this means that extra blocks can be added in a modular fashion to improve the frequency response. See [14] for a discussion of implementation issues and block diagrams.

To illustrate the maximally flat low-pass differentiator, Fig. 1 shows the frequency response for a family of Type IV low-pass differentiators of length $N = 30$, where K is varied from 0 to 24 in increments of 4, and where $L = (N - K)/2 - 1$. The bandwidth of the frequency response depends on the relative values of K and L. When $K = 0$ we obtain a full band differentiator. Similarly Fig. 2 shows the frequency response for a family of Type III low-pass differentiators of length $N = 31$, where K is varied from 1 to 25 in increments of 4, and where $L = (N - K)/2 - 1$.

V. Conclusion

This paper describes the design of low-pass linear-phase FIR digital differentiators according to the maximally flat criterion. The solutions can not be obtained from a low-pass filter as in the case of a full-band differentiator. The algorithms for automatic sum simplification described in [15] were used to obtain a simple two-term recurrence relation for computing the coefficients of the impulse response. Equations (58) and (60) contain the main result.

There are several possible extensions to the problem described in this paper. For example, the extension of the recursive formulas to the case where the maximally flat approximation to the ideal differentiator is performed not at $\omega = 0$ but at another frequency ω_o. For the full-band differentiator, solutions are given in [1], [11], and [12]. This type of solution is relevant when the signal is centered around a known frequency (as in radar using Doppler tracking [12]).

Another remaining question is the existence of low-complexity structures for maximally flat differentiators, of the kind described in [22], [23] by Samadi, Nishihara, and Iwakura for maximally flat low-pass filters. Those structures are multiplierless and have a regular structure.

Another extension of the approach described in this paper is the design of second (and higher) order differentiators as in [21] where the desired frequency response is $-\omega^2$, etc. For applications where a variable fractional sample delay is required, an extension along the lines of [5] will be of interest.

Appendix A
Derivation of $c_2(n)$

To find $c_2(n)$ in (34), note that

$$A_2'(x) = \frac{\dfrac{1}{\sqrt{1-x}} - \dfrac{1}{2}\dfrac{\arccos(1-2x)}{\sqrt{x}}}{x} \tag{61}$$

$$= \frac{\dfrac{1}{\sqrt{1-x}} - \dfrac{1}{2}A_2(x)}{x} \tag{62}$$

so $A_2(x)$ satisfies the differential equation

$$A_2'(x)\cdot x + \frac{1}{2}A_2(x) = \frac{1}{\sqrt{1-x}}. \tag{63}$$

Substituting the Taylor series for $A_2(x)$ into this differential equation gives

$$\sum_{n=0}^{\infty} c_2(n)\left(n+\frac{1}{2}\right)x^n = \frac{1}{\sqrt{1-x}}. \tag{64}$$

Now note that

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} g(n)x^n \tag{65}$$

where

$$g(n) = \frac{1}{n!}\cdot\left(\frac{1}{2}\right)_n \tag{66}$$

and where $(a)_n$ is the Pochhammer function, or rising factorial, defined as

$$(a)_n = \underbrace{(a)(a+1)\ldots(a+n-1)}_{n\ \text{terms}}. \tag{67}$$

Then we can write

$$\sum_{n=0}^{\infty} c_2(n)\left(n+\frac{1}{2}\right)x^n = \sum_{n=0}^{\infty} g(n)x^n. \tag{68}$$

Therefore, matching like powers in (68), we get

$$c_2(n) = \frac{g(n)}{n + 1/2} \tag{69}$$

or

$$c_2(n) = \frac{2}{2n+1} \cdot \frac{1}{n!} \cdot \left(\frac{1}{2}\right)_n. \tag{70}$$

REFERENCES

[1] J. Le Bihan, "Maximally linear FIR digital differentiators," *J. Circuits, Syst., Signal Processing*, vol. 14, pp. 633–637, May 1995.

[2] ——, "Coefficients of FIR digital differentiators and Hilbert transformers for midband frequencies," *IEEE Trans. Circuits Syst. II*, vol. 43, pp. 272–274, Mar. 1996.

[3] B. Carlsson, "Maximum flat digital differentiator," *Electron. Lett.*, vol. 27, no. 8, pp. 675–677, Apr. 11, 1991.

[4] DSP Committee, *Selected Papers in Digital Signal Processing, II*. New York: IEEE Press, 1976.

[5] E. Hermanowicz and M. Rojewski, "Design of FIR first order digital differentiators of variable fractional sample delay using maximally flat error criterion," *Electron. Lett.*, vol. 30, no. 1, pp. 17–18, Jan. 6, 1994.

[6] O. Herrmann, "On the approximation problem in nonrecursive digital filter design," *IEEE Trans. Circuit Theory*, vol. 18, pp. 411–413, May 1971.

[7] I. R. Khan and R. Ohba, "New design of full band differentiators based on Taylor series," in *Proc. Inst. Elect. Eng.—Visual Image Signal Process*, vol. 146, Aug. 1999, pp. 185–189.

[8] B. Kumar and S. C. D. Roy, "Coefficients of maximally linear, FIR digital differentiators for low frequencies," *Electron. Lett.*, vol. 24, no. 9, pp. 563–565, Apr. 28, 1988.

[9] ——, "Design of digital differentiators for low frequencies," *Proc. IEEE*, vol. 76, no. 3, pp. 287–289, Mar. 1988.

[10] ——, "Design of efficient FIR digital differentiators and Hilbert transformers for midband frequency ranges," *Int. J. Circuit Theory Appl.*, vol. 17, pp. 483–488, 1989.

[11] ——, "Maximally linear FIR digital differentiators for midband frequencies," *Int. J. Circuit Theory Appl.*, vol. 17, pp. 21–27, 1989.

[12] B. Kumar, S. C. D. Roy, and H. Shah, "On the design of FIR digital differentiators which are maximally linear at the frequency π/p, $p \in \{\text{positive integers}\}$," *IEEE Trans. Signal Processing*, vol. 40, pp. 2334–2338, Sept. 1992.

[13] T. W. Parks and C. S. Burrus, *Digital Filter Design*. New York: Wiley, 1987.

[14] S.-C. Pei and P.-H. Wang, "Closed-form design of maximally flat FIR Hilbert transformers, differentiators, and fractional delayers by power series expansion," *IEEE Trans. Circuits Syst. I*, vol. 48, pp. 389–389, Apr. 2001.

[15] M. Petkovšek, H. S. Wilf, and D. Zeilberger, $A = B$. Natick, MA: A.K. Peters, 1996. [Online]. Available: http://www.cis.upenn.edu/~wilf/AeqB.html.

[16] W. Preuss, , NY: Symbol Technologies, 2000. Personal communication.

[17] R. Rabenstein, "Design of FIR digital filters with flatness constraints for the error function," *Circuits, Syst., Signal Processing*, vol. 13, pp. 77–97, Jan. 1993.

[18] L. R. Rabiner and B. Gold, *Theory and Application of Digital Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1975.

[19] L. R. Rabiner, J. H. McClellan, and T. W. Parks, "FIR digital filter design techniques using weighted Chebyshev approximation," *Proc. IEEE*, vol. 63, pp. 595–610, Apr. 1975.

[20] L. R. Rabiner and C. M. Rader, Eds., *Digital Signal Processing*. New York: IEEE Press, 1972.

[21] M. R. R. Reddy, S. C. D. Roy, and B. Kumar, "Design of efficient second and higher degree FIR digital differentiators for midband frequencies," in *Proc. Inst. Elect. Eng.*, vol. 138, Feb. 1991, pp. 29–33.

[22] S. Samadi, T. Cooklev, A. Nishihara, and N. Fujii, "Multiplierless structure for maximally flat linear phase FIR filters," *Electron. Lett.*, vol. 29, no. 2, pp. 184–185, Jan. 21, 1993.

[23] S. Samadi, A. Nishihara, and H. Iwakura, "Universal maximally flat low-pass FIR systems," *IEEE Trans. Signal Processing*, vol. 48, pp. 1956–1964, July 2000.

[24] H. W. Schüssler and P. Steffen, "Some advanced topics in filter design," in *Advanced Topics in Signal Processing*, J. S. Lim and A. V. Oppenheim, Eds. Engelwood Cliffs, NJ: Prentice-Hall, 1988, ch. 8, pp. 416–491.

[25] I. W. Selesnick and C. S. Burrus, "Exchange algorithms for the design of linear phase FIR filters and differentiators having flat monotonic passbands and equiripple stopbands," *IEEE Trans. Circuits Syst. II*, vol. 43, pp. 671–675, Sept. 1996.

On the Low-Power Implementation of FIR Filtering Structures on Single Multiplier DSPs

Ahmet Teyfik Erdogan and Tughrul Arslan

***Abstract*—The authors present three multiplication schemes for the low-power implementation of finite-impulse response (FIR) filters on single multiplier complementary metal–oxide–semiconductor (CMOS) digital signal processors (DSPs). The schemes achieve power reduction through the minimization of switching activity at one or both inputs of the multiplier. In addition, these schemes are characterized by their flexibility since they tradeoff implementation cost against power consumption. Results are provided for a number of example FIR filters demonstrating power savings ranging from 20% with schemes which can be implemented on existing common DSPs, and up to 51% with schemes using enhanced DSP architectures.**

***Index Terms*—Digital signal processors (DSPs), finite-impulse response (FIR) filters, low power design, switching activity.**

I. INTRODUCTION

The advent of portable computing has led to a significant increase in research work targeting the reduction of power consumption in high throughput digital signal processor (DSP) devices. It can be shown that the most significant factor affecting power consumption in a complementary metal–oxide–semiconductor (CMOS) very large-scale integration (VLSI) device is the switching power, which is expressed by the product $[1/2 \times (\text{supply voltage})^2 \times \text{switched capacitance} \times f]$ [1]. In this product, the switched capacitance is a combination of the physical capacitance, C, and the switching activity factor, k. This factor is defined as the average number of times that a gate makes a logic transition ($1 \rightarrow 0$ or $0 \rightarrow 1$) in each clock cycle. Therefore, for achieving low power in CMOS circuits one must target minimizing one or more of the parameters V_{dd}, C and k.

Recently a number of researchers have targeted the minimization of switching activity with the aim of reducing power consumption for generic digital circuits, e.g., [2] and [3]. For example, in [2] a synthesis system is developed which synthesises both finite state machines and combinational logic for low-power applications. Low power is achieved by minimizing the average number of transitions at the internal nodes of the combinational circuits and state assignment which minimizes the total number of transitions occurring at the

Manuscript received September 1, 1999; revised June 1, 2001. This paper was recommended by Associate Editor J. LeBlanc.

The authors are with the Department of Electronics and Electrical Engineering, University of Edinburgh, Edinburgh EH9 3JL, Scotland, U.K. (e-mail: ate@ee.ed.ac.uk, arslan@ee.ed.ac.uk).

Publisher Item Identifier S 1057-7130(02)06229-8.