Maximally Flat Lowpass Digital Differentiators

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Abstract

This paper describes the design of Type III and Type IV linear-phase FIR lowpass digital differentiators according to the maximally flat criterion. We introduce a two-term recursive formula that enables the simple stable computation of the impulse response coefficients. The same recursive formula is valid for both Type III and Type IV solutions.

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1 Introduction

This paper describes the design of linear-phase FIR lowpass digital differentiators according to the maximally flat criterion. We introduce a two-term recursive formula that enables the simple stable computation of the impulse response coefficients. The same recursive formula is valid for both Type III and Type IV solutions.

To avoid the undesirable amplification of noise in digital differentiation, lowpass differentiators can be used in place of full-band ones. They find application, for example, in bar code readers [16]. Maximally flat digital full-band differentiators have been described by several authors [3, 7, 8, 9, 14] where the approximation is accurate at a single point, usually $\omega = 0, \pi/2$, or π. When the input signal lies in a known frequency band centered around the frequency ω_o, it is desirable to use a differentiator which is maximally flat at ω_o rather than at $\omega = 0$. Solutions of that type have been introduced in [2, 10, 11, 12]. The design of lowpass differentiators can be performed with least squares, the Remez algorithm [18] and other methods [13, 23, 24]; however, the design of maximally flat lowpass differentiators has not been previously addressed.

The frequency response of the ideal (full-band) digital differentiator is

$$H_{FB}(e^{j\omega}) = j\,\omega, \quad |\omega| < \pi. \tag{1}$$

The maximally flat FIR approximation to the ideal differentiator satisfies the derivative constraints,

$$|H(e^{j\omega})| = 0, \quad \omega = 0 \tag{2}$$

$$\frac{d}{d\omega}|H(e^{j\omega})| = 1, \quad \omega = 0 \tag{3}$$

$$\frac{d^k}{d\omega^k}|H(e^{j\omega})| = 0, \quad \omega = 0, \quad 2 \le k \le B-1 \tag{4}$$

where B is the number of free parameters. Notice that the approximation is performed at a single frequency point $\omega = 0$. Kumar and Dutta Roy described how the solution can be obtained from the maximally flat lowpass FIR filter [9].

The frequency response of the ideal lowpass digital differentiator is

$$H_{LP}(e^{j\omega}) = \begin{cases} j\,\omega & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| < \pi \end{cases} \tag{5}$$

where ω_c is the cutoff frequency. This paper describes the maximally flat linear-phase FIR approximation, which satisfies the constraints

$$|H(e^{j\omega})| = 0, \quad \omega = 0 \tag{6}$$

$$\frac{d}{d\omega}|H(e^{j\omega})| = 1, \quad \omega = 0 \tag{7}$$

$$\frac{d^k}{d\omega^k}|H(e^{j\omega})| = 0, \quad \omega = 0, \quad 2 \le k \le 2L \tag{8}$$

$$\frac{d^k}{d\omega^k}|H(e^{j\omega})| = 0, \quad \omega = \pi, \quad 0 \le k \le 2M. \tag{9}$$

In this case, it is not a single point approximation problem and the solution can not be obtained directly from the maximally flat lowpass filter. Type III and Type IV linear-phase FIR solutions to this problem are derived in Section 2 and Section 3 respectively. As in [14], the method used in this paper is based on power series expansion. To compute the impulse response coefficients in a simple way, Section 4 gives a two-term recursive formula which is valid for both Type III and IV solutions. This recursive formula was obtained using the algorithms of [15] for automatic hypergeometric-type sum simplification. A Matlab program to compute the maximally flat differentiator based on the two-term recurrence is available from the author. The derivation of the maximally flat lowpass differentiator below uses the transformation of variables used, for example, by Herrmann in the derivation of maximally flat lowpass Type I FIR filters [6]. Similar to [14], the expression obtained in (58) below for the transfer function can be used as the basis of a structure for an efficient implementation of the filter.

2 Type IV Solution

The derivation of the solution will depend on a transformation that maps polynomials on the real interval $[0, 1]$ to polynomials on the upper half of the unit circle. Let $P(x)$ be a polynomial,

$$P(x) = p_0 + p_1\, x + p_2\, x^2 + \cdots + p_n\, x^n. \tag{10}$$

If we define

$$H(z) = P\left(\frac{-z + 2 - z^{-1}}{4}\right), \tag{11}$$

then $H(z)$ will be a Type I transfer function and its frequency response is given by

$$H(e^{j\omega}) = P\left(\frac{1}{2} - \frac{1}{2}\cos(\omega)\right). \tag{12}$$

Now, note that a type IV transfer function $H_4(z)$ can always be written as

$$H_4(z) = \frac{1}{2}\,(1 - z^{-1})\,H(z). \tag{13}$$

(This is because a Type IV transfer function always has a zero at $z = 1$. See [13, 17], for example, for a description of the four types of FIR linear-phase digital filter.) The frequency response of $H_4(z)$ is given by

$$\begin{aligned}
H_4(e^{j\omega}) &= \frac{1}{2}\,(1 - e^{-j\omega})\,H(e^{j\omega}) && (14)\\
&= j \cdot e^{-j\frac{\omega}{2}}\ \sin\left(\frac{\omega}{2}\right)\ H(e^{j\omega}) && (15)\\
&= j \cdot e^{-j\frac{\omega}{2}}\ \sqrt{\frac{1}{2} - \frac{1}{2}\cos(\omega)} \cdot P\left(\frac{1}{2} - \frac{1}{2}\cos(\omega)\right) && (16)\\
&= j \cdot e^{-j\frac{\omega}{2}}\ P_4\left(\frac{1}{2} - \frac{1}{2}\cos(\omega)\right) && (17)
\end{aligned}$$

where

$$P_4(x) := \sqrt{x}\,P(x). \tag{18}$$

Note that even though the term $e^{-j\frac{\omega}{2}}$ appears in the expression for $H_4(e^{j\omega})$ no fractional delay element is needed for the implementation of this digital filter. ($H(z)$ is a Type I transfer function which requires no fractional delays, and (13) does not introduce any fractional delay.)

Our goal is to design $P(x)$ so that $H_4(e^{j\omega})$ approximates the ideal lowpass differentiator. That means it is desired that

$$P_4\left(\frac{1}{2} - \frac{1}{2}\cos(\omega)\right) \approx \begin{cases} \omega & \text{for } \omega \approx 0 \\ 0 & \text{for } \omega \approx \pi. \end{cases} \tag{19}$$

Equivalently,

$$P_4(x) \approx \begin{cases} \arccos(1 - 2x) & \text{for } x \approx 0 \\ 0 & \text{for } x \approx 1 \end{cases} \tag{20}$$

or in terms of $P(x)$,

$$P(x) \approx \begin{cases} \dfrac{\arccos(1 - 2x)}{\sqrt{x}} & \text{for } x \approx 0 \\ 0 & \text{for } x \approx 1. \end{cases} \tag{21}$$

These approximation condition can be written as

$$P(x) = A(x) \cdot (1 - x)^M \tag{22}$$

and

$$\frac{\arccos(1-2x)}{\sqrt{x}} - P(x) = B(x) \cdot x^{L+1} \tag{23}$$

where M, and $L+1$ are the degrees of tangency at $x=1$ and $x=0$. The parameters M and L are the parameters that define the solution. $A(x)$ and $B(x)$ are power series in x. Adding (22) and (23), we get

$$\frac{\arccos(1-2x)}{\sqrt{x}} = B(x) \cdot x^{L+1} + A(x) \cdot (1-x)^M. \tag{24}$$

Solving for $A(x)$ we get,

$$A(x) = \frac{1}{(1-x)^M} \cdot \left(\frac{\arccos(1-2x)}{\sqrt{x}} - B(x) \cdot x^{L+1} \right). \tag{25}$$

Let the Taylor series of the first term be given by:

$$\frac{1}{(1-x)^M} \cdot \frac{\arccos(1-2x)}{\sqrt{x}} = \sum_{n=0}^{\infty} c(n)\, x^n. \tag{26}$$

Note that the polynomial $A(x)$ of minimal degree which satisfies (25) will be of degree L. (Suppose the degree of $A(x)$ were less than L, then it would be impossible to choose a power series $B(x)$ so that the right-hand side of (25) has the same degree as the left-hand side.) With $A(x)$ being minimal degree, we have

$$A(x) = \sum_{n=0}^{L} c(n)\, x^n, \tag{27}$$

$$P(x) = (1-x)^M \sum_{n=0}^{L} c(n)\, x^n, \tag{28}$$

and

$$\begin{aligned} H_4(z) &= \frac{1}{2}\,(1-z^{-1})\, H(z) &(29)\\ &= \frac{1}{2}\,(1-z^{-1})\, P\left(\frac{-z+2-z^{-1}}{4}\right) &(30)\\ &= \frac{1}{2}\,(1-z^{-1}) \left(\frac{z+2+z^{-1}}{4}\right)^M \sum_{n=0}^{L} c(n) \left(\frac{-z+2-z^{-1}}{4}\right)^n. &(31) \end{aligned}$$

Note that (1) the degree of $P(x)$ is $M+L$, (2) the degree of $H_4(z)$ is $2M+2L+1$. That makes $h_4(n)$ an impulse response of length $2M+2L+2$. We need only find the Taylor series coefficients $c(n)$. To this end, we define

$$A_1(x) := \frac{1}{(1-x)^M} = \sum_{n=0}^{\infty} c_1(n)\, x^n \tag{32}$$

where

$$c_1(n) = \binom{M+n-1}{n} \tag{33}$$

and

$$A_2(x) := \frac{\arccos(1-2x)}{\sqrt{x}} = \sum_{n=0}^{\infty} c_2(n)\, x^n \tag{34}$$

where

$$c_2(n) = \frac{2}{2\,n+1} \cdot \frac{1}{n!} \cdot \left(\frac{1}{2}\right)_n \tag{35}$$

which is derived in the Appendix. Then $A(x) = A_1(x)\, A_2(x)$ and

$$c(n) = \sum_{k=0}^{n} c_1(k)\, c_2(n-k). \tag{36}$$

In Section 4, we give a recursive formula for computing the coefficients $c(n)$.

3 Type III Solution

Note that a type III transfer function $H_3(z)$ can always be written as

$$H_3(z) = \frac{1}{2}\,(1-z^{-1})\,\frac{1}{2}\,(1+z^{-1})\,H(z) \tag{37}$$

where $H(z)$ is a Type I transfer function. (This is because a Type III transfer function always has a zero at $z = 1$ and $z = -1$.) The frequency response of $H_3(z)$ is given by

$$\begin{aligned} H_3(e^{j\omega}) &= j \cdot e^{-j\frac{\omega}{2}}\ \sin\left(\frac{\omega}{2}\right)\ e^{-j\frac{\omega}{2}}\ \cos\left(\frac{\omega}{2}\right)\ H(e^{j\omega}) && (38)\\ &= j \cdot e^{-j\omega}\ \sqrt{\frac{1}{2} - \frac{1}{2}\cos(\omega)} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\cos(\omega)} \cdot P\left(\frac{1}{2} - \frac{1}{2}\cos(\omega)\right) && (39)\\ &= j \cdot e^{-j\frac{\omega}{2}}\ P_3\left(\frac{1}{2} - \frac{1}{2}\cos(\omega)\right) && (40) \end{aligned}$$

where

$$P_3(x) := \sqrt{x}\,\sqrt{1-x}\,P(x). \tag{41}$$

To design $P(x)$ so that $H_3(e^{j\omega})$ approximates the ideal lowpass differentiator, we ask that

$$P_3\left(\frac{1}{2} - \frac{1}{2}\cos(\omega)\right) \approx \begin{cases} \omega & \text{for } \omega \approx 0 \\ 0 & \text{for } \omega \approx \pi. \end{cases} \tag{42}$$

Equivalently,

$$P_3(x) \approx \begin{cases} \arccos(1-2x) & \text{for } x \approx 0 \\ 0 & \text{for } x \approx 1 \end{cases} \tag{43}$$

or in terms of $P(x)$,

$$P(x) \approx \begin{cases} \dfrac{\arccos(1-2x)}{\sqrt{x}\,\sqrt{1-x}} & \text{for } x \approx 0 \\ 0 & \text{for } x \approx 1. \end{cases} \tag{44}$$

These approximation condition can be written as

$$P(x) = A(x) \cdot (1-x)^M \tag{45}$$

and

$$\frac{\arccos(1-2x)}{\sqrt{x}\,\sqrt{1-x}} - P(x) = B(x) \cdot x^{L+1} \tag{46}$$

where M, and $L+1$ are the degrees of tangency at $x=1$ and $x=0$. Adding (45) and (46), we get

$$\frac{\arccos(1-2x)}{\sqrt{x}\sqrt{1-x}} = B(x) \cdot x^{L+1} + A(x) \cdot (1-x)^M. \tag{47}$$

As above, the polynomial $A(x)$ of minimal degree which satisfies (47) will be of degree L. Solving for $A(x)$ of minimal degree L we get

$$A(x) = \sum_{n=0}^{L} c(n)\, x^n \tag{48}$$

where $c(n)$ are the coefficients in the Taylor series of

$$\frac{\arccos(1-2x)}{\sqrt{x}\,(1-x)^{M+\frac{1}{2}}} = \sum_{n=0}^{\infty} c(n)\, x^n. \tag{49}$$

Then

$$H_3(z) = \frac{1}{2}(1-z^{-1})\,\frac{1}{2}(1+z^{-1})\,H(z) \tag{50}$$

$$= \frac{1}{4}(1-z^{-1})(1+z^{-1})\,P\left(\frac{-z+2-z^{-1}}{4}\right) \tag{51}$$

$$= \frac{1}{4}(1-z^{-1})(1+z^{-1})\left(\frac{z+2+z^{-1}}{4}\right)^M \sum_{n=0}^{L} c(n)\left(\frac{-z+2-z^{-1}}{4}\right)^n. \tag{52}$$

Note that (1) $P(x)$ is of degree $M+L$ again, and (2) $H_3(z)$ is of degree $2M+2L+2$. That makes $h_3(n)$ an impulse response of length $2M+2L+3$. To determine $c(n)$, define $A_1(x)$ as

$$A_1(x) := \frac{1}{(1-x)^{M+\frac{1}{2}}} = \sum_{n=0}^{\infty} c_1(n)\, x^n \tag{53}$$

where

$$c_1(n) = \binom{M+\frac{1}{2}+n-1}{n} \tag{54}$$

and define $A_2(x)$ as in (34). Then $A(x) = A_1(x)\,A_2(x)$ and $c(n) = \sum_{k=0}^{n} c_1(k)\, c_2(n-k)$. The coefficients $c_2(n)$ are the same for Type III and Type IV solutions, while $c_1(n)$ has a different form.

To evaluate the binomial coefficient for fractional values of the upper entry, we can use the Gamma function Γ,

$$c_1(n) = \binom{M+\frac{1}{2}+n-1}{n} = \frac{\Gamma(M+\frac{1}{2}+n)}{\Gamma(M+\frac{1}{2})\,\Gamma(n+1)}. \tag{55}$$

The Gamma function also satisfies $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$, so we can write a recursive equation for evaluating $c_1(n)$,

$$c_1(n) = \frac{1}{n}\ (M+n-1/2)\cdot c_1(n-1). \tag{56}$$

In Section 4 we unify the Type III and Type IV solutions.

4 Combined Formula

Let K denote the number of zeros a transfer function has at $z=-1$. A Type IV transfer function always has an even number of zeros at $z=-1$; for the solution given in Section 3 we have $K=2M$. On the other hand, a Type III transfer function always has an odd number of zeros at $z=-1$; for the solution given in Section 2 we have $K=2M+1$. In terms of K, the coefficients $c_1(n)$ in (33) and (54) are given by the same formula, namely,

$$c_1(n) = \binom{K/2+n-1}{n}. \tag{57}$$

A causal maximally flat lowpass differentiator with linear phase is given by

$$H(z) = \left(\frac{1-z^{-1}}{2}\right)\left(\frac{1+z^{-1}}{2}\right)^K z^{-L} \sum_{n=0}^{L} c(n)\ \left(\frac{-z+2-z^{-1}}{4}\right)^n \tag{58}$$

Table 1: Numerical values of the weights $c(n)$ for $0 \le K \le 3$, $0 \le n \le 10$.

	$c(n)$			
n	$K=0$	$K=1$	$K=2$	$K=3$
0	2.0000	2.0000	2.0000	2.0000
1	0.3333	1.3333	2.3333	3.3333
2	0.1500	1.0667	2.4833	4.4000
3	0.0893	0.9143	2.5726	5.3143
4	0.0608	0.8127	2.6334	6.1270
5	0.0447	0.7388	2.6781	6.8658
6	0.0347	0.6820	2.7128	7.5478
7	0.0279	0.6365	2.7408	8.1843
8	0.0231	0.5991	2.7639	8.7834
9	0.0195	0.5675	2.7834	9.3509
10	0.0168	0.5405	2.8002	9.8914

with

$$c(n) = \sum_{k=0}^{n} c_1(k)\, c_2(n-k) \tag{59}$$

where $c_1(n)$ is given by (57) and $c_2(n)$ is given by (35). Using the algorithms given in [15] we can find the following two-term recursive formula for computing the coefficients $c(n)$.

$$c(n) = \frac{(8n^2 + 4Kn - 10n - K + 3)\, c(n-1) - (2n + K - 3)^2\, c(n-2)}{2n\,(2n+1)} \tag{60}$$

for $n \ge 2$ with $c(0) = 2$, and $c(1) = K + 1/3$. The formula in (58) with the recursion for $c(n)$ in (60) generates both Type III and Type IV solutions. When K is even we obtain a Type IV transfer function, when K is odd we obtain a Type III transfer function. In either case, the length of the impulse response is $N = K + 2L + 2$. Table 1 gives numerical values for $c(n)$ for several values of K and n. Notice that $c(n)$ does not depend on L; rather L determines how many values of $c(n)$ are needed in (58). As noted in [14], this means that extra blocks can be added in a modular fashion to improve the frequency response. See [14] for a discussion of implementation issues and block diagrams.

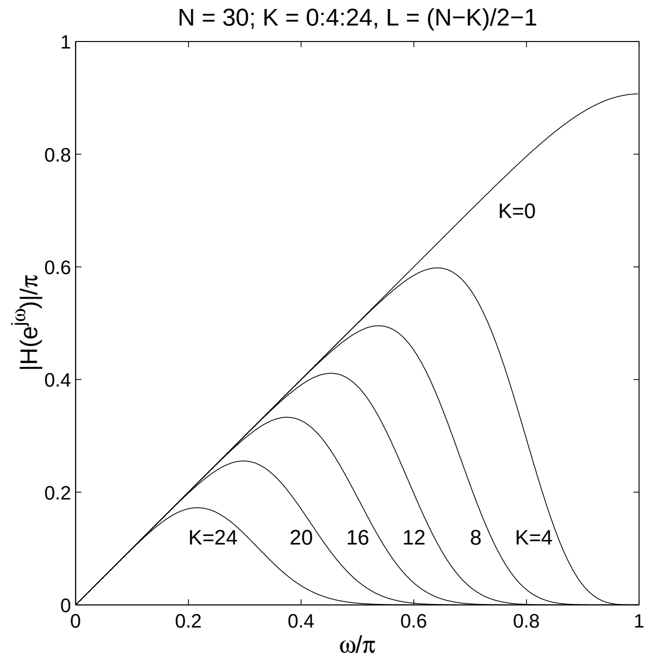

Figure 1: Type IV maximally flat lowpass differentiators.

To illustrate the maximally flat lowpass differentiator, Figure 1 shows the frequency response for a family of Type IV lowpass differentiators of length $N = 30$, where K is varied from 0 to 24 in increments of 4, and where $L = (N - K)/2 - 1$. The bandwidth of the frequency response depends on the relative values of K and L. When $K = 0$ we obtain a full band differentiator. Similarly Figure 2 shows the frequency response for a family of Type III lowpass differentiators of length $N = 31$, where K is varied from 1 to 25 in increments of 4, and where $L = (N - K)/2 - 1$.

5 Conclusion

This paper describes the design of lowpass linear-phase FIR digital differentiators according to the maximally flat criterion. The solutions can not be obtained from a lowpass filter as in the case of a full-band differentiator. The algorithms for automatic sum simplification described in [15] were used to obtain a simple two-term recurrence relation for computing the coefficients of the impulse response. The equations (58) and (60) contain the main result.

There are several possible extensions to the problem described in this paper. For example, the extension of the recursive formulas to the case where the maximally flat approximation to the ideal differentiator is performed not at $\omega = 0$ but at another frequency ω_o. For the full-band

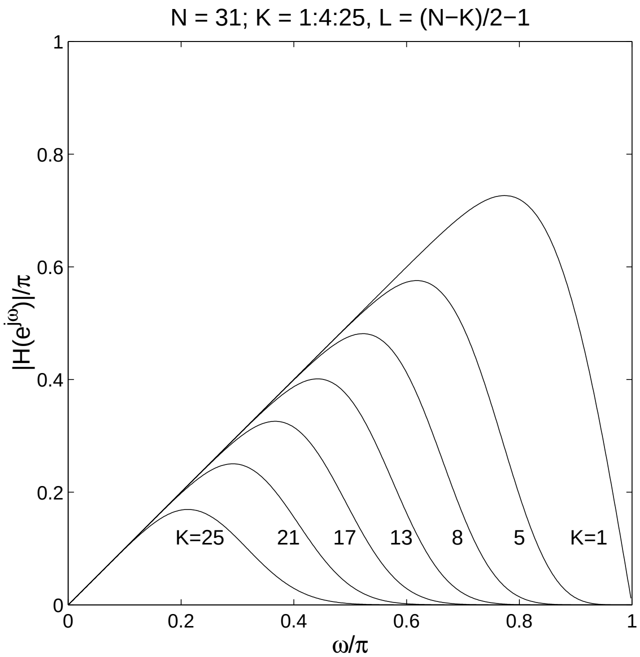

Figure 2: Type III maximally flat lowpass differentiators.

differentiator, solutions are given in [1, 11, 12]. This type of solution is relevant when the signal is centered around a known frequency (as in radar using Doppler tracking [12]).

Another remaining question is the existence of low-complexity structures for maximally flat differentiators, of the kind described in [21, 22] by Samadi, Nishihara, and Iwakura for maximally flat lowpass filters. Those structures are multiplierless and have a regular structure.

Another extension of the approach described in this paper is the design of second (and higher) order differentiators as in [20] where the desired frequency response is $-\omega^2$, etc. For applications where a variable fractional sample delay is required, an extension along the lines of [5] will be of interest.

A Derivation of $c_2(n)$

To find $c_2(n)$ in (34), note that

$$A_2'(x) = \frac{\dfrac{1}{\sqrt{1-x}} - \dfrac{1}{2}\dfrac{\arccos(1-2x)}{\sqrt{x}}}{x} \tag{61}$$

$$= \frac{\dfrac{1}{\sqrt{1-x}} - \dfrac{1}{2}A_2(x)}{x} \tag{62}$$

so $A_2(x)$ satisfies the differential equation

$$A_2'(x)\cdot x + \frac{1}{2}\,A_2(x) = \frac{1}{\sqrt{1-x}}. \tag{63}$$

Substituting the Taylor series for $A_2(x)$ into this differential equation gives

$$\sum_{n=0}^{\infty} c_2(n)\,(n+\frac{1}{2})\,x^n = \frac{1}{\sqrt{1-x}}. \tag{64}$$

Now note that

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} g(n)\,x^n \tag{65}$$

where

$$g(n) = \frac{1}{n!}\cdot\left(\frac{1}{2}\right)_n \tag{66}$$

and where $(a)_n$ is the Pochhammer function, or rising factorial, defined as

$$(a)_n = \underbrace{(a)\,(a+1)\,\cdots\,(a+n-1)}_{n\text{ terms}}. \tag{67}$$

Then we can write

$$\sum_{n=0}^{\infty} c_2(n)\,(n+\frac{1}{2})\,x^n = \sum_{n=0}^{\infty} g(n)\,x^n. \tag{68}$$

Therefore, matching like powers in (68), we get

$$c_2(n) = \frac{g(n)}{n+1/2} \tag{69}$$

or

$$c_2(n) = \frac{2}{2\,n+1}\cdot\frac{1}{n!}\cdot\left(\frac{1}{2}\right)_n. \tag{70}$$

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