# Lowpass Filters Realizable as Allpass Sums: Design via a New Flat Delay Filter*

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**Abstract**

This paper describes a new class of maximally flat lowpass recursive digital filters. The filters are realizable as a parallel sum of two allpass filters, a structure for which low-complexity low-noise implementations exist. Note that, with the classical Butterworth filter of degree $N$, which is retrieved as a special case, it is not possible to adjust the delay (or phase-linearity). However, with the more general class of filters described in this paper, the adjustment of the delay becomes possible, and the trade-off between the delay and the phase-linearity can be chosen. The construction of these lowpass filters depends upon a new maximally flat delay allpole filter, for which the degrees of flatness at $\omega = 0$ and $\omega = \pi$ are not necessarily equal. For the coefficients of this flat delay filter, an explicit solution is introduced, which also specializes to a previously known result.

## 1 Introduction

The design of digital allpass filters has received much attention in recent years, for it has become well known that $(i)$ low complexity structures with low roundoff noise behavior are available for allpass filters [32, 43], and $(ii)$ they are useful components in a variety of applications. Indeed, while the traditional applications of allpass filters appear to be phase equalization [7, 19] and fractional delay elements [18], their uses in multirate filtering, filterbanks, notch filtering, recursive phase splitters and Hilbert transformers have also

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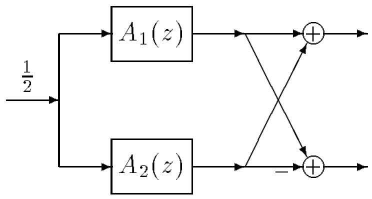


Figure 1: Allpass sum. The upper branch gives $\frac{1}{2}[A_1(z)+A_2(z)]$. The lower branch gives the complementary filter $\frac{1}{2}[A_1(z)-A_2(z)]$ without additional filtering.

been described [30, 35, 38, 37]. Of particular interest has been the design of frequency selective filters realizable as a parallel combination of two allpasses,

$$H(z) = \frac{1}{2}\left[A_1(z) + A_2(z)\right], \tag{1}$$

illustrated in Figure 1. This structure, the allpass sum, has its history in analog lattice circuitry [45], and wave digital filters [9].

It is interesting to note that digital filters of odd degree, obtained from the classical analog (Butterworth, Chebyshev, and elliptic) prototypes via the bilinear transformation, can be realized as allpass sums[1] [34, 43, 44, 42]. As allpass sums, such filters can be realized with low complexity structures that are robust to finite precision effects [43]. However, the realization of a lowpass filter as an allpass sum is not restricted to the classical filter prototypes. The allpass sum is a useful generalization of the classical transfer functions, that is honored with a number of benefits. Specifically, when some degree of phase linearity is desired, non-classical filters of the form (1) can be designed that achieve superior results with respect to implementation complexity and phase linearity [11, 14].

The desired degree of phase linearity can, in fact, be structurally incorporated. If one of the allpass branches in an allpass sum contains only delay elements, then the allpass sum exhibits approximately linear phase in the passbands [17, 31]. The frequency selectivity is then obtained by appropriately designing the remaining allpass branch. Interestingly, by varying the number of delay elements used and the degrees of $A_1(z)$ and $A_2(z)$, the phase linearity can be affected. Simultaneous approximation of the phase and magnitude is a difficult problem in general, so the ability to structurally incorporate this aspect of the approximation problem is attractive.

While general procedures for allpass design [4, 10, 15, 16, 21, 24, 26, 27, 28, 36, 46] are applicable to the design of frequency selective allpass sums, several publications have addressed, in addition to the general problem, the details specific to allpass sums [1, 2, 13, 22, 31, 33, 35]. Several authors has also described iterative Remez-like exchange algorithms for the design of allpass filters and allpass sums according to the Chebyshev criterion [11, 14, 20, 38].

[1] The classical filters of even degree can also be realized as a sum of two allpass filters, however, in that case complex coefficients are required. In this paper, only real-coefficient filters will be considered.

This paper considers the design of maximally flat lowpass filters realizable as allpass sums. It is explained that the design problem can be formulated as the problem of group delay approximation of a single allpole filter[2]. The solution sought in this paper depends on an allpole filter, the group delay of which possesses a maximally flat characteristic at $\omega = 0$ and $\omega = \pi$. An explicit solution is given for this allpole filter, where the degrees of flatness at $\omega = 0$ and $\omega = \pi$ need not be equal. Also, by developing a method by which the cut-off frequency can be continuously varied, it is found that the classical Butterworth digital filter is retrieved a special case of the class of lowpass filters described in this paper. Various examples are provided and simple Matlab programs are given in the appendix. The solution given is applicable to the approximately linear phase case, and the general case.

# 2 Flat Delay Filter

Let $G(\omega)$ denote the group delay, the negative derivative of the phase with respect to $\omega$. This section examines transfer function

$$\frac{b_0}{\sum_{n=0}^{N} a_n z^{-n}} = \frac{b_0}{D(z)} \tag{2}$$

where $a_0 = 1$ and the DC gain is normalized to unity: $b_0 = \sum_{n=0}^{N} a_n$. The goal is to design the transfer function so that its group delay possesses a specified degree of flatness at $\omega = 0$ and $\omega = \pi$, where the degree of flatness at each point is not necessarily equal. There are $N$ free parameters $a_1, \ldots, a_N$. $K$ will denote the number of constraints assigned to $\omega = 0$, and $L$ will denote the number of constraints assigned to $\omega = \pi$. The problem is formulated as follows:

Given $\tau$, (the desired group delay), $K$, $L$, and $N$, with $K + L = N$, find $N$ real coefficients $a_n$ such that the group delay of (2) satisfies the derivative constraints

1. $G(\omega = 0) = \tau$. (If $K > 0$).
2. $G(\omega = \pi) = \tau$. (If $L > 0$).
3. $G^{(2i)}(\omega = 0) = 0$ for $i = 1, \ldots, K - 1$.
4. $G^{(2i)}(\omega = \pi) = 0$ for $i = 1, \ldots, L - 1$.

Because the group delay is an even function of $\omega$, the odd indexed derivatives of $G(\omega)$ are automatically zero, so they do not need to be specified. The solution has the property that $G^{(i)}(\omega = 0) = 0$ for $i = 1, \ldots, 2K - 1$, and $G^{(i)}(\omega = \pi) = 0$ for $i = 1, \ldots, 2L - 1$.

[2] In this paper, the term "allpole filter" allows for zeros at $z = 0$

Following the derivation given in [41], the four group delay conditions (1.-4.) are equivalent to the following system of equations, linear in the coefficients $a_n$:

$$\sum_{n=1}^{N}(n+\tau)^{2k+1}a_n = -\tau^{2k+1} \qquad \text{for } k = 0, 1, \ldots, K-1 \tag{3}$$

$$\sum_{n=1}^{N}(-1)^n(n+\tau)^{2l+1}a_n = -\tau^{2l+1} \qquad \text{for } l = 0, 1, \ldots, L-1. \tag{4}$$

The system matrix is comprised of two Vandermonde-like matrices, corresponding to (3) and (4) respectively. Vandermonde matrices arise in polynomial interpolation problems, and accordingly, become ill conditioned in cases where many derivatives at a few points are specified, as is the case here. With some assistance from the computer algebra system, Maple [6], the solution was found to be

$$a_n = \frac{(-1)^n}{(2\tau+K+L+1)_n}\binom{K+L}{n}\sum_{i=0}^{L}(-4)^i\binom{L}{i}\frac{(\tau)_i(n-i+1)_i(2\tau+2i)_{n-i}}{(K+L+1-i)_i} \tag{5}$$

for $0 \leq n \leq N$. The Pochhammer symbol $(x)_n$ denotes the rising factorial: $(x)_n = (x)\cdot(x+1)\cdot(x+2)\cdots(x+n-1)$. We define $(x)_n$ as 1 for $n \leq 0$. Although (5) looks rather cumbersome, the coefficients $a_n$ can be efficiently calculated. A method for doing so, is described in the appendix, where a Matlab program for this calculation is also given.

Note that for some negative values of $\tau$ no solution exists. These are values for which the denominator in (5) vanishes: $\tau = -\frac{1}{2}(K+L+n)$ for $n = 1, \ldots, N$. These values of $\tau$ are more negative than will be needed later on. That there should be $N$ values of $\tau$ for which no solution exists is expected, since the determinant of the system matrix is a polynomial in $\tau$ of degree $N$. Except for those $N$ values, $\tau$ can be any real number, positive or negative — although $1/D(z)$ will be unstable for sufficiently negative values of $\tau$.

The use of (5) for negative values of $\tau$ is not purely for theoretical interest. In fact, for the design of stable causal lowpass filters realizable as allpass sums, the use of unstable solutions (5) obtained with negative values of $\tau$ will be necessary, as illustrated in Section 3. Properties of the zeros of $1/D(z)$, that will be relevant later on, are discussed in Appendix A.

**Example 1:** To illustrate the maximally flat delay filter, an ninth degree example is provided. For this example, $N = 9$, $K = 6$ and $L = 3$. The coefficients $a_n$ as rational functions of $\tau$ are given in Table 1, as well as the solution for $\tau = 7/2$ and $\tau = -3/2$. The group delay and the pole-zero diagram for those two filters are shown in Figures 2 and 3. For $\tau = 7/2$, the solution is stable. For $\tau = -3/2$, the solution possesses two poles outside the unit circle.

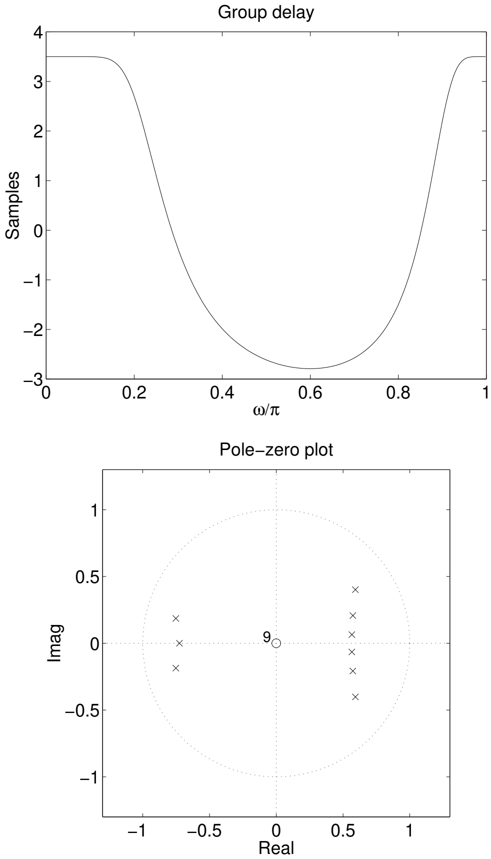


Figure 2: Maximally flat delay filter, $K = 6$, $L = 3$, $\tau = 7/2$.

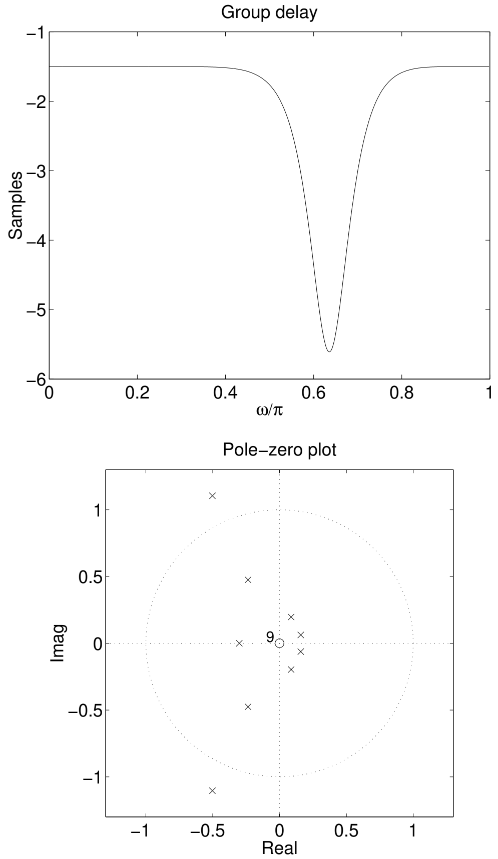


Figure 3: Maximally flat delay filter, $K = 6$, $L = 3$, $\tau = -3/2$.

Table 1: Maximally flat delay digital allpole filter, $K = 6$, $L = 3$, $\tau = 7/2$ and $\tau = -3/2$.

| $n$ | $a_n$ | $a_n, (\tau = 7/2)$ | $a_n, (\tau = -3/2)$ |
|---|---|---|---|
| 0 | $1$ | $1$ | $1$ |
| 1 | $-\frac{3\,\tau}{5+\tau}$ | $-\frac{21}{17}$ | $\frac{9}{7}$ |
| 2 | $-\frac{36\,\tau}{(2\,\tau+11)(5+\tau)}$ | $-\frac{14}{17}$ | $\frac{27}{14}$ |
| 3 | $\frac{(16\,\tau+116)(1+\tau)\tau}{(2\,\tau+11)(5+\tau)(6+\tau)}$ | $\frac{602}{323}$ | $\frac{23}{42}$ |
| 4 | $-\frac{(6+6\,\tau)(2\,\tau-3)\tau}{(2\,\tau+11)(5+\tau)(6+\tau)}$ | $-\frac{84}{323}$ | $\frac{3}{14}$ |
| 5 | $-\frac{6\,\tau\,(2\,\tau+21)(2+\tau)(1+\tau)}{(5+\tau)(2\,\tau+11)(7+\tau)(6+\tau)}$ | $-\frac{308}{323}$ | $-\frac{9}{154}$ |
| 6 | $\frac{(16\,\tau+28)(2+\tau)(1+\tau)\tau}{(5+\tau)(2\,\tau+11)(7+\tau)(6+\tau)}$ | $\frac{154}{323}$ | $\frac{1}{462}$ |
| 7 | $\frac{(108+36\,\tau)(2+\tau)(1+\tau)\tau}{(2\,\tau+11)(8+\tau)(7+\tau)(6+\tau)(5+\tau)}$ | $\frac{858}{7429}$ | $\frac{9}{2002}$ |
| 8 | $-\frac{3\,\tau\,(2\,\tau+7)(3+\tau)(2+\tau)(1+\tau)}{(2\,\tau+11)(8+\tau)(7+\tau)(6+\tau)(5+\tau)}$ | $-\frac{1001}{7429}$ | $-\frac{3}{2002}$ |
| 9 | $\frac{\tau\,(2\,\tau+7)(4+\tau)(3+\tau)(2+\tau)(1+\tau)}{(2\,\tau+11)(9+\tau)(8+\tau)(7+\tau)(6+\tau)(5+\tau)}$ | $\frac{1001}{37145}$ | $\frac{1}{6006}$ |

### 2.1 A Special Case

The allpole filter possessing a flat delay characteristic only at DC ($L = 0$) is well established in the literature. In this case, the solution (5) specializes to the previously reported solution:

$$a_n = (-1)^n \binom{N}{n} \frac{(2\tau)_n}{(2\tau + N + 1)_n}. \tag{6}$$

This solution has been described in various forms, see for example [8, 12, 40, 41]. For example, in [41] this formula is written as

$$a_n = (-1)^n \binom{N}{n} \prod_{i=0}^{N} \frac{2\tau + i}{2\tau + n + i} \tag{7}$$

which is the same as (6) after simplification (cancelling the $N + 1 - n$ terms common to the numerator and denominator of the $N + 1$ term product). It should also be noted that the solution of [40] is the same when it is understood that [40] considers the transfer function $z^{-N}/D(z)$; the appropriate change of variables is straightforward.

Various uses of the special case (6) have been described. One application of (6) is the design of allpass fractional delay filters [18]. It is also used in one of several approaches described in [23] for the design of negative group delay systems. It should also be noted that, if one sets $K = L$ in (5), then $D(z)$ is an even function of $z$. In this case one can write $D(z) = E(z^2)$, where $E(z)$ can be obtained using (6). Hence, with the substitution $z \rightarrow z^2$, the special case '$L = 0$' is mapped to the special case '$K = L$'. That solution is also useful for the design of halfband filters [37] and has been developed and used for the design of biorthogonal filterbanks and wavelet bases in [29] (see also [3]).

It should be noted that the bilinear transformation (BLT) does not preserve the flat delay characteristic. Therefore, the Bessel filter (the analog filter with a maximally flat delay characteristic at DC) does not provide, via the BLT, the digital filter (6) having that characteristic.

# 3 The Allpass Sum

It will be useful to write the allpass sum as

$$\frac{1}{2}\left[z^{-d}A_2(z) + A_1(z)\right] \tag{8}$$

where $A_1(z)$ and $A_2(z)$ are both stable causal allpasses and $d$ is a nonnegative integer. Note that the allpass sum

$$\frac{1}{2}\left[z^{-d} + \frac{A_1(z)}{A_2(z)}\right] \tag{9}$$

has the same magnitude response as (8), but has a different phase response and has poles outside the unit circle. In the design of allpass sums for magnitude approximation, it is therefore sufficient, as noted in [11], to consider only the design of an allpass sum of the form

$$\frac{1}{2}\left[z^{-d} + A(z)\right] \tag{10}$$

where $A(z)$ may possess poles outside the unit circle in general. The factorization of $A(z)$ into $A_1(z)/A_2(z)$, where $A_1(z)$ and $A_2(z)$ are both stable causal allpasses, is straightforward. With such a factorization, the transfer function (8) will be stable causal. For this reason, stability of $A(z)$ is unimportant during the design phase — a causal stable lowpass filter of the form (8) can be simply obtained even when $A(z)$ is unstable. If $A(z)$ is itself stable causal, then the total response has approximately linear phase, as previously mentioned. The degree of phase linearity of the total response depends, in part, on the relative degrees of $A_1(z)$ and $A_2(z)$. For greater values of $d$ the phase will be more linear but the overall delay will be greater. As will be illustrated by examples, this trade-off between phase linearity and overall delay can be controlled by appropriate selection of $d$ and design of $A(z)$.

Consider the design of an allpass sum of the form (10) where $A(z)$ is an allpass filter of degree $N$ with real coefficients. Being allpass, $A(z)$ is of the form

$$A(z) = \frac{z^{-N}D(1/z)}{D(z)} \tag{11}$$

where $D(z)$ is a real polynomial in $z^{-1}$ of degree $N$. In addition, if $A(z)$ satisfies the approximations

1. $A(z) \approx z^{-d}$ in the passband

2. $A(z) \approx -z^{-d}$ in the stopband

then the total response of (10) will clearly be lowpass. It follows that the group delay of $A(z)$ should approximate $d$ in both the passband and the stopband. Consequently, the group delay of the transfer function $1/D(z)$ should approximate $(d-N)/2$ in both the passband and the stopband. (Note that if the group delay of $1/D(z)$ is $\tau$, then the group delay of $A(z) = z^{-N}D(1/z)/D(z)$ is $2\tau + N$. Therefore, if the group delay of $A(z)$ is to be $X$, then the group delay of $1/D(z)$ is to be $(X-N)/2$. Various useful expressions for the group delay are provided in [34].)

The solution sought in this paper depends upon the transfer function $1/D(z)$, the group delay of which approximates $(d-N)/2$ in the maximally flat sense. This solution is obtained by setting derivatives of the group delay equal to zero at $\omega = 0$ and $\omega = \pi$, and was introduced in Section 2. The lowpass filters described herein are maximally flat in the following sense. Given the allpass sum structure (8) where $d$ and $N$ are specified, the coefficients are chosen so that as many derivatives of the magnitude response as possible vanish at $\omega = 0$ and $\omega = \pi$. As in Section 2, $K$ will denote the number of constraints assigned to $\omega = 0$, and $L$ will denote the number of constraints assigned to $\omega = \pi$. The relative values of $K$ and $L$ determine the location of the transition region. As above $N = K + L$.

Some restrictions apply to the selection of $d$, $K$ and $L$. Leaving the details in appendix B, $d$, $K$ and $L$ must be selected so that they satisfy the following two conditions.

1. $|K - L| + 1 \leq d \leq K + L + 1$.

2. $d$ must have the same parity as $K + L + 1$.

It follows that $\tau = (d-N)/2$ must be an odd multiple of one half. Also, when $K$ and $L$ are approximately equal, more choices for $d$ are available, than when $K$ and $L$ are disparate. Note that, for fixed values of $K$ and $L$, $d$ can take on $(K + L - |K - L|)/2 + 1$ different values. Similarly, for a fixed value of $d$, there are $d$ different permissible $(K, L)$ pairs.

As noted previously, the allpass $A(z)$ will in general possess poles lying outside the unit circle — thus requiring the factorization $A(z) = A_1(z)/A_2(z)$ where $A_1(z)$ and $A_2(z)$ are both stable causal. The degree of $A_2(z)$ is equal to the number of poles of $A(z)$ outside the unit circle, while the degree of $A_1(z)$ is $N$ minus this number. Denote the degrees of $A_1(z)$ and $A_2(z)$ by $n_1$ and $n_2$ respectively. From the observations in appendix A, one has

$$n_2 = 2\left\lfloor \frac{N-d+1}{4} \right\rfloor \tag{12}$$

$$n_1 = N - 2\left\lfloor \frac{N-d+1}{4} \right\rfloor . \tag{13}$$

The important special case $d = N - 1$, yields a lowpass filter with approximately linear phase at $\omega = 0$; for $d = N - 1$, one has $n_2 = 0$ and no factorization of $A(z)$ is necessary. In this case $H(z) = (z^{-d} + A(z))/2$ is stable causal. Although the same is true for $d = N+1$, the corresponding lowpass filters have a significantly poorer magnitude response.

To be explicit, the stable causal $H(z)$ is obtained as follows. Let $z_i$, for $i = 1, \ldots, N$, denote the zeros of $D(z)$. Then $D_1(z)$ and $D_2(z)$ are respectively formed from the zeros of $D(z)$ that lie inside and outside the unit circle:

$$D_1(z) = \prod_{|z_i|<1} (z^{-1} - z_i^{-1}) \tag{14}$$

$$D_2(z) = \prod_{|z_i|>1} (z^{-1} - z_i). \tag{15}$$

Then the allpass subfilters are given by

$$A_1(z) = \frac{z^{-n_1} D_1(1/z)}{D_1(z)} \tag{16}$$

$$A_2(z) = \frac{z^{-n_2} D_2(1/z)}{D_2(z)}. \tag{17}$$

Define $\tilde{D_1}(z) = z^{-n_1} D_1(1/z)$ and $\tilde{D_2}(z) = z^{-n_2} D_2(1/z)$. The allpass sum is then

$$H(z) = \frac{1}{2}\left[z^{-d}\frac{\tilde{D_2}(z)}{D_2(z)} + \frac{\tilde{D_1}(z)}{D_1(z)}\right] \tag{18}$$

$$= \frac{1}{2}\left[\frac{z^{-d}\tilde{D_2}(z)D_1(z) + \tilde{D_1}(z)D_2(z)}{D_2(z)D_1(z)}\right]. \tag{19}$$

From (19), it is evident that the numerator of $H(z)$ is symmetric and of degree $N+d$. It is worth noting that $H(z)$ can be implemented as an allpass sum with $N$ multiplications per sample and $N+d$ delay elements. (An allpass of degree $n$ can be implemented with $n$ multiplications per sample and $n$ delay elements [25].)

**Example 2:** To illustrate the design of a lowpass filter realizable as an allpass sum, consider an example where $K = 6$, $L = 3$, $(N = 9)$ and $d = 6$. Then $\tau = -3/2$, and the allpass $A(z) = z^{-N}D(1/z)/D(z)$ is to be formed from the flat delay filter $1/D(z)$ shown in Figure 3. The unwrapped phase responses of $z^{-d}$ and $A(z)$ are shown in Figure 4. Note that the unwrapped phase of $A(z)$ takes on the value $(d-1)\pi$ at $\omega = \pi$. The lowpass behavior of their sum follows from this behavior of the phase difference. However, as $A(z)$ has two poles outside the unit circle, the factorization $A(z) = A_1(z)/A_2(z)$, where $A_1(z)$ and $A_2(z)$ are stable causal, is necessary. In this example, $A_1(z)$ is of degree $n_1 = 7$ and $A_2(z)$ is of degree $n_2 = 2$. The unwrapped phase responses of $z^{-d}A_2(z)$ and $A_1(z)$ are shown in Figure 4, where the phase difference remains unchanged. The magnitude response, group delay, and pole-zero plot of $(z^{-d}A_2(z) + A_1(z))/2$ are shown in Figure 5. Note that some of the poles and zeros inside the unit circle almost cancel, a behavior which has been noted before [11].

We have formed lowpass filters from flat delay filters whose group delay derivatives vanish at $\omega = 0$ and $\omega = \pi$, but does it follow that the lowpass filters obtained in this way have magnitude response derivatives that vanish at $\omega = 0$ and $\omega = \pi$? In the following, this is shown to be the case.

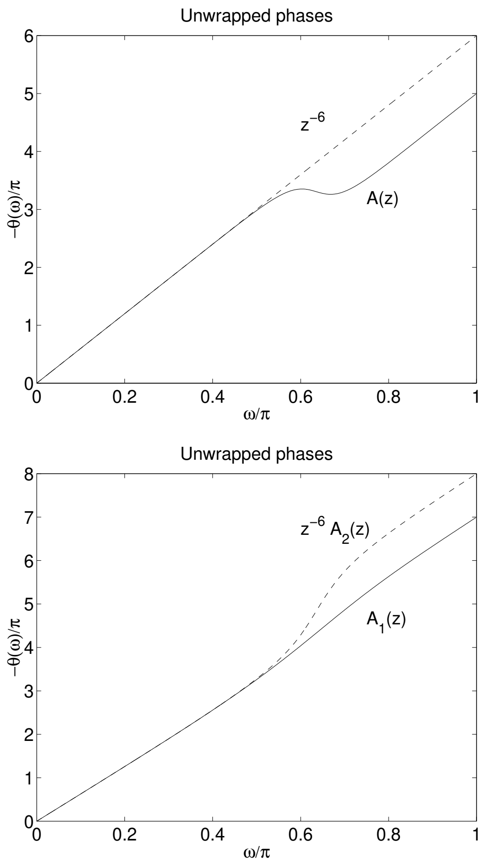


Figure 4: The unwrapped phase responses of $z^{-6}$ and $A(z)$ gives rise to the desired lowpass behavior of the sum $(z^{-6}+A(z))/2$. The factorization $A(z) = A_1(z)/A_2(z)$ into stable causal allpasses yields a stable causal filter $(z^{-6}A_2(z) + A_1(z))/2$ with the same magnitude response.

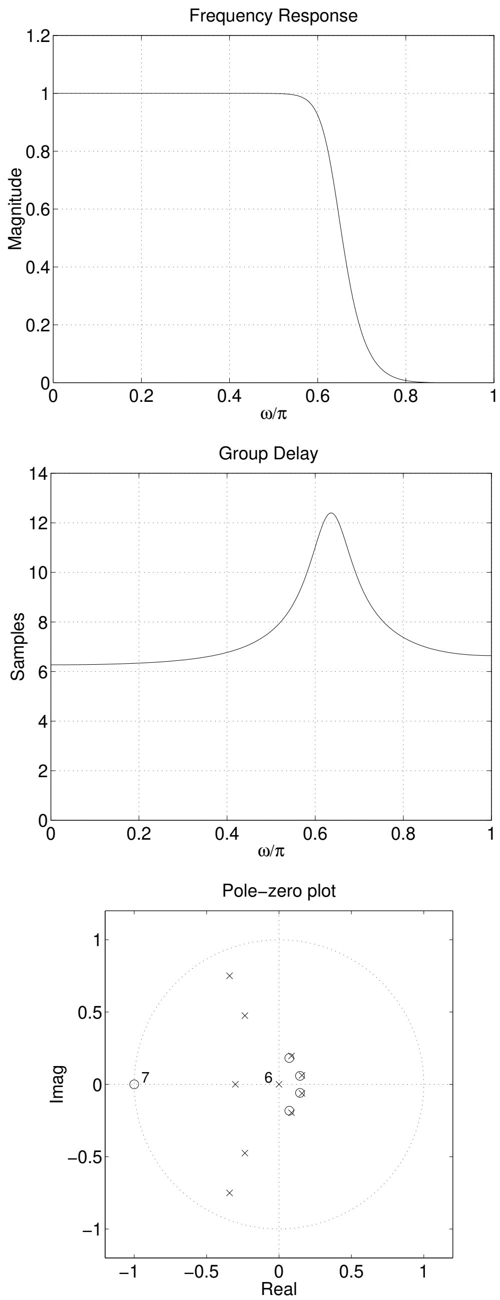


Figure 5: Sum of two allpass filters, $K = 6$, $L = 3$, $d = 6$. The zeros that lie outside the unit circle are not shown. Since the numerator is symmetric real, for each zero $z_i$ inside the unit circle, another zero lies at $1/z_i$.

Beginning with the transfer function,

$$H(z) = \frac{1}{2}(z^{-d} + A(z))$$

where $A(e^{j\omega}) = e^{j\theta(\omega)}$, the square magnitude response of $H(z)$ will be denoted by $F(\omega)$

$$\begin{aligned} F(\omega) &= |H(e^{j\omega})|^2 && (20)\\ &= H(e^{j\omega})H(e^{-j\omega}) && (21)\\ &= \frac{1}{4}(e^{-jd\omega} + e^{j\theta(\omega)})(e^{jd\omega} + e^{-j\theta(\omega)}) && (22)\\ &= \frac{1}{2} + \frac{1}{2}\cos(d\omega + \theta(\omega)). && (23) \end{aligned}$$

Consider first the point $\omega = 0$. With the flat behavior of $G(\omega)$, the group delay of $A(\omega)$, at $\omega = 0$, $G(\omega)$ can be written as $G(\omega) = d + c_0\,\omega^{2K} + O(\omega^{2K+1})$. Therefore $\theta(\omega) = -d\,\omega + c\,\omega^{2K+1} + O(\omega^{2K+2})$ near $\omega = 0$. Note that $1/2 + \cos(x)/2 = 1 - x^2/4 + O(x^4)$ near $x = 0$, and hence

$$F(\omega) = 1 - \frac{1}{4}c^2\omega^{4K+2} + O(\omega^{4K+3})$$

near $\omega = 0$. Therefore $F^{(i)}(\omega = 0) = 0$ for $i = 1, \ldots, 4K+1$.

Now consider the point $\omega = \pi$. With the flat behavior of $G(\omega)$ at $\omega = \pi$, $G(\omega)$ can be written as $G(\omega) = d + c_0\,(\omega - \pi)^{2L} + O((\omega-\pi)^{2L+1})$ near $\omega = \pi$. Therefore $\theta(\omega) = c_2 - d\,\omega + c\,(\omega-\pi)^{2L+1} + O((\omega-\pi)^{2L+2})$. Because $d\,\pi + \theta(\pi)$ must be an odd multiple of $\pi$ (so that $F(\pi) = 0$), $c_2$ is equal to $l\pi$ where $l$ is some odd integer. Note that $1/2 + \cos(x)/2 = (x - l\pi)^2/4 + O((x - l\pi)^4)$ near $x = l\pi$, hence,

$$F(\omega) = \frac{1}{4}c^2(\omega - \pi)^{4L+2} + O((\omega - \pi)^{4L+3})$$

near $\omega = \pi$. Therefore $F^{(i)}(\omega = \pi) = 0$ for $i = 0, \ldots, 4L+1$. It follows that the transfer function $H(z)$ in (18) possesses a zero at $z = -1$ of multiplicity $2L + 1$;

The flatness properties of $F(\omega)$ correspond to the flatness properties of $G(\omega)$. Summarizing, with $F(\omega) = |H(e^{j\omega})|^2$,

$$\begin{aligned} F(\omega = 0) &= 1 && (24)\\ F^{(i)}(\omega = 0) &= 0 \quad \text{for } i = 1, \ldots, 4K+1. && (25)\\ F^{(i)}(\omega = \pi) &= 0 \quad \text{for } i = 0, \ldots, 4L+1. && (26) \end{aligned}$$

**Example 3:** By varying $K$ and $L$, while keeping $N = K + L$ and $d$ constant, the location of the cut-off frequency can be varied. For $N = 9$, $d = 8$, Figure 6 illustrates the solutions obtained by varying $K$ from 1 to 8.

**Example 4:** By varying $d$, while keeping $K$ and $L$ constant, the delay response of $H(z)$ can be varied. For $K = 5$ and $L = 4$, Figure 7 illustrates the solutions obtained by varying $d$ from 2 to 8 in increments of 2.

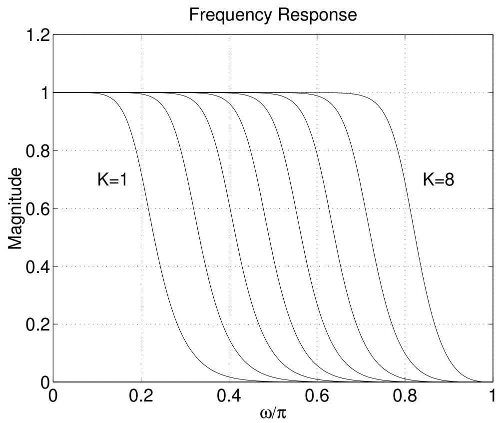


Figure 6: Sum of two allpass filters. $N = 9$, $d = 8$, $K$ is varied from 1 to 8, $L = N - K$. (It follows that $\tau = -\frac{1}{2}$).

Note that for larger values of $d$, the overall delay is greater, but that the phase is more linear. Note that the filters in Figure 7 do not all posses the same cut-off frequency $\omega_c$. That is to be expected because $\omega_c$ is not part of the problem formulated. The cut-off frequency is determined by the parameters $K$ and $L$. However, in Section 4, it will be shown how to fine tune $\omega_c$ so that the filters achieve prescribed values of $\omega_c$.

The design method does suffer from the drawback that there is no simple formula for selecting $K$, $L$, and $d$ to obtain a filter having a prescribed cut-off frequency $\omega_c$. However the relationship between these three parameters and the filter characteristics, such as $\omega_c$ and the DC group delay $G(0)$, can be investigated by plotting $(\omega_c, G(0))$ as a coordinate in a plane. In this way, the qualitative behavior of $\omega_c$ and $G(0)$ as a function of $K$,$L$, and $d$ can be examined.

For $N = K + L = 9$, Figure 8 illustrates this for various lowpass filters $H(z)$ obtained from degree 9 allpole filters. The parameters $(d, K, L)$ are indicated in the figure, for $N = K + L = 9$. In the figure, the frequency $\omega_c$ is that frequency at which $|H(e^{j\,\omega})| = \frac{1}{2}$.

# 4 Continuous Variation of the Cut-off Frequency

For a fixed value of $d$, the design approach described above produces a discrete set of $d$ filters. For these filters, the location of the cut-off frequency is entirely determined by $d$, $K$, and $L$. However, by giving up a single derivative constraint, the location of the cut-off frequency can be continuously varied. In this way, the cut-off frequency can be precisely specified, if desired. The method begins with the flat delay filter described in Section 2. Let us make the dependence of $a_n$ on $K$ and $L$ in (5) explicit by the notation $a_n(K, L)$. Similarly, we will use the notation $D(z; K, L)$, etc. The ability to vary the cut-off frequency is obtained by

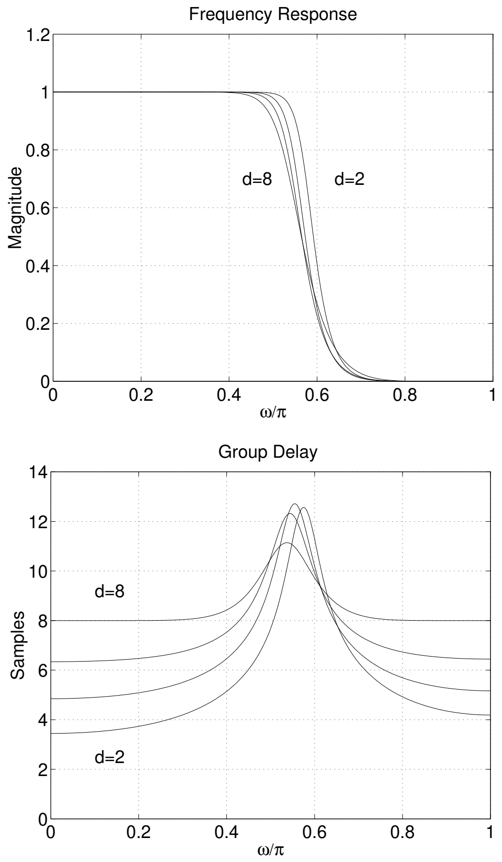


Figure 7: Sum of two allpass filters, $K = 5$, $L = 4$, $d$ is varied from 2 to 8 in increments of 2. (It follows that $\tau$ is varied from $-3.5$ to $-0.5$ in increments of 1.)

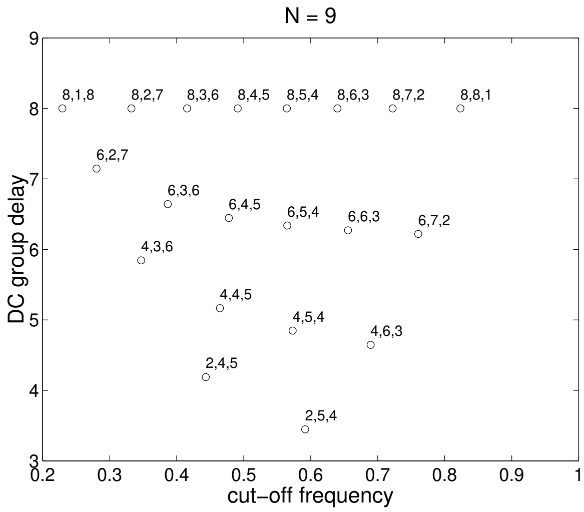


Figure 8: Locations in the $\omega_c$-$G(0)$ plane of maximally flat allpass sums (8) for which the sum of the degrees of $A_1(z)$ and $A_2(z)$ is 9. The parameters $(d, K, L)$ are indicated in the figure.

forming a weighted average of two filters:

$$D(z;K,L,\alpha) = \alpha \cdot D(z;K,L+1) + (1-\alpha)\cdot D(z;K+1,L) \tag{27}$$

$$= \sum_{n=0}^{K+L+1} (\alpha \cdot a_n(K,L+1) + (1-\alpha)\cdot a_n(K+1,L))z^{-1} \tag{28}$$

where $\alpha$ lies in the real interval $[0,1]$. The group delay of $1/D(z;K,L,\alpha)$ has a flat characteristic at $\omega = 0$ and $\omega = \pi$, but the total number of contiguous vanishing derivatives is one less than the maximum number achievable. The reason this *linear* combination of coefficients retains the flatness parameters $K$ and $L$, is that the equations (3), (4) for flatness are linear. Note that $D(z;K,L,\alpha)$ is of degree $K+L+1$.

By continuously varying $\alpha$ in $[0,1]$, the cut-off frequency can be continuously varied. The allpass $A(z)$ and the lowpass $H(z)$ are now written as

$$A(z;K,L,\alpha) = \frac{z^{-(K+L+1)}D(1/z;K,L,\alpha)}{D(z;K,L,\alpha)} \tag{29}$$

$$H(z;\alpha) = \frac{1}{2}(z^{-d} + A(z;K,L,\alpha)). \tag{30}$$

Certainly there is no simple functional form for the way in which the zeros vary with $\alpha$. Therefore, the formation of $D_1(z)$ and $D_2(z)$ for a specific value of $\alpha$ requires computing the zeros of $D(z;K,L,\alpha)$ for that value.

When a weighted average is used as described in this section, the conditions $d$, $K$, and $L$ must satisfy, given in Section 3, are to be modified. Because one of the derivative conditions is given up, the conditions become somewhat looser. The less restrictive conditions on $d$, $K$ and $L$ are:

1. $|K-L| \leq d \leq K+L+2$.

2. $d$ must have the same parity as $K+L$.

Note that when $K$ and $L$ are chosen to be equal, $d$ may be selected to be 0. In this case, the weighted average filter is the classical digital Butterworth filter of odd degree. Hence, the filters described in this paper can be interpreted as a generalization of the Butterworth filter. Note that the generalization of the Butterworth filter described in this paper, is quite different from the generalization described in [39]. The filters described in this paper are realizable as allpass sums, while those in [39] did not have that constraint imposed.

**Example 5:** For an example, with $K=3$, $L=5$, and $d=8$, we obtain a degree 9 allpass filter. For $\alpha = 0.5$, the magnitude response of the allpass sum is shown as a dashed line in Figure 9.

Figure 10 illustrates some of the achievable points in the $\omega_c$-$G(0)$ plane when $A(z)$ is of degree 9, obtained by continuously varying the cut-off frequency $\omega_c$. This figure indicates how to choose the parameters $(d,K,L)$

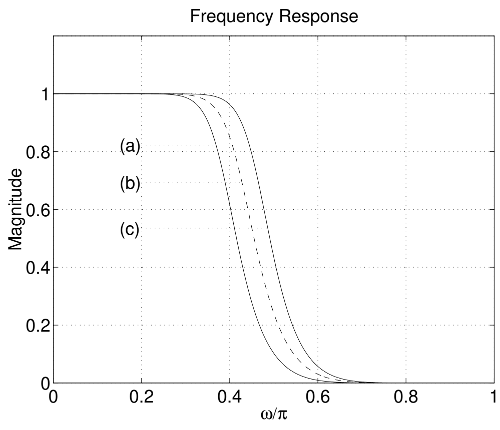


Figure 9: Variation of cut-off frequency. For each of the filters shown, $d = 8$, and $N$, the degree of the allpass filter $A(z)$, is 9. The flatness parameters of the three filters are: (a) $K = 3$, $L = 6$, (b) $K = 3$, $L = 5$, (c) $K = 4$, $L = 5$. The filter shown as a dashed line, (b), is obtained by using a weighted average as described in the text with $\alpha = 0.5$.

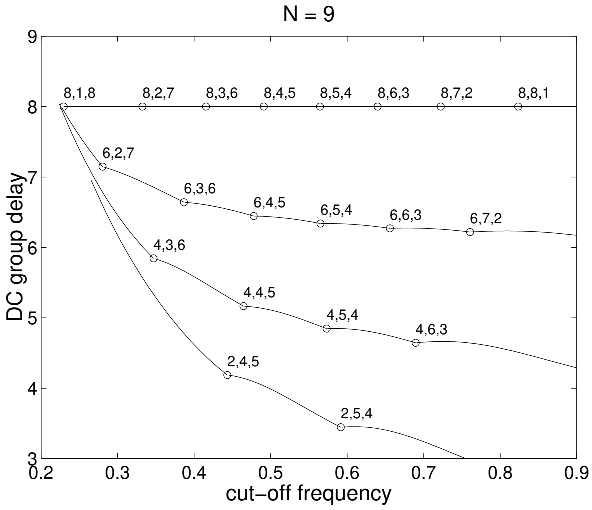


Figure 10: Locations in the $\omega_c$-$G(0)$ plane of allpass sums (8) for which the sum of the degrees of $A_1(z)$ and $A_2(z)$ is 9, obtained by the continuous variation of the the cut-off frequency $\omega_c$.

to obtain a filter having a desired bandwidth and delay. It is interesting to note that at the vertices, the curves are slightly cusped.

How does one chose $\alpha$ to satisfy a prescribed cut-off frequency $\omega_c$? First, one should check that the desired cut-off frequency $\omega_c$ lies between the cut-off frequencies of $H(z;\alpha=0)$ and $H(z;\alpha=1)$. Otherwise, the following formula for $\alpha$ can not be expected to yield a desirable solution. Given $\omega_c$, the method to calculate the appropriate value $\alpha$ can be derived by beginning with the equation $|H(e^{j\omega_c})| = \frac{1}{2}$. Using the expression for the square magnitude (23), one then gets $\frac{1}{2}+\frac{1}{2}\cos(d\,\omega_c+\theta(\omega_c)) = \frac{1}{4}$ where $A(e^{j\omega}) = e^{j\theta(\omega)}$. Using (11) one obtains $\theta(\omega) = -2\beta(\omega) - N\omega$ where $D(e^{j\omega}) = |D(e^{j\omega})|e^{j\beta(\omega)}$. In turn, this gives,

$$\frac{1}{2}+\frac{1}{2}\cos(d\,\omega_c - 2\beta(\omega_c) - N\omega_c) = \frac{1}{4}.$$

Solving for $\beta(\omega_c)$, the desired value of the phase of $D(z;K,L,\alpha)$ at $z=e^{j\omega_c}$, gives

$$\beta(\omega_c) = \pm\frac{\pi}{3} - \frac{\omega_c}{2} + l\,\pi \tag{31}$$

where $l$ is some integer. From the desired phase $\beta(\omega_c)$, the desired value of $\alpha$ can be determined. Let us define $r_0$, $r_1$, $\beta_0$, $\beta_1$ by

$$D(e^{j\omega_c};K+1,L) =: r_0 e^{j\beta_0} \tag{32}$$

$$D(e^{j\omega_c};K,L+1) =: r_1 e^{j\beta_1}. \tag{33}$$

Then $D(z=e^{j\omega_c};K,L,\alpha)$ can be written as $\alpha\, r_1\, e^{j\beta_1} + (1-\alpha)\, r_0\, e^{j\beta_0}$. The sought value of $\alpha$ is such that

$$\alpha\, r_1\, e^{j\beta_1} + (1-\alpha)\, r_0\, e^{j\beta_0} = X\cdot e^{j\beta_c}$$

where $X$ is some positive real number and $\beta_c$ is the desired phase (31). After some geometry, the solution is found to be

$$\alpha = \frac{r_0\sin(\beta_0-\beta_1) - r_0\cos(\beta_0-\beta_1)\tan(\beta_c-\beta_1)}{r_0\sin(\beta_0-\beta_1) + (r_1 - r_0\cos(\beta_0-\beta_1))\tan(\beta_c-\beta_1)}$$

where $\beta_c = \pm\frac{\pi}{3} - \frac{\omega_c}{2}$. The additive multiple of $\pi$ in (31) can be dropped because tangent is periodic with $\pi$. This leaves two candidate values of $\alpha$. It was found in practice, that only one of those two values will lie in $[0,1]$.

**Example 6:** Figure 11 illustrates a set lowpass filters, where the $\omega_c$ is prescribed to be $0.56\pi$. Compare to Example 4. The filter with least delay is the classical Butterworth filter. The figure shows that with this approach the trade-off, between the delay and the phase linearity, can be managed with out affecting the magnitude response significantly.

# 5 Conclusion

This paper presents two main results. $(i)$ The design and closed form solution of an allpole filter (or allpass filter), the group delay of which approximates any value with prescribed flatness $K$ and $L$ at $\omega=0$ and

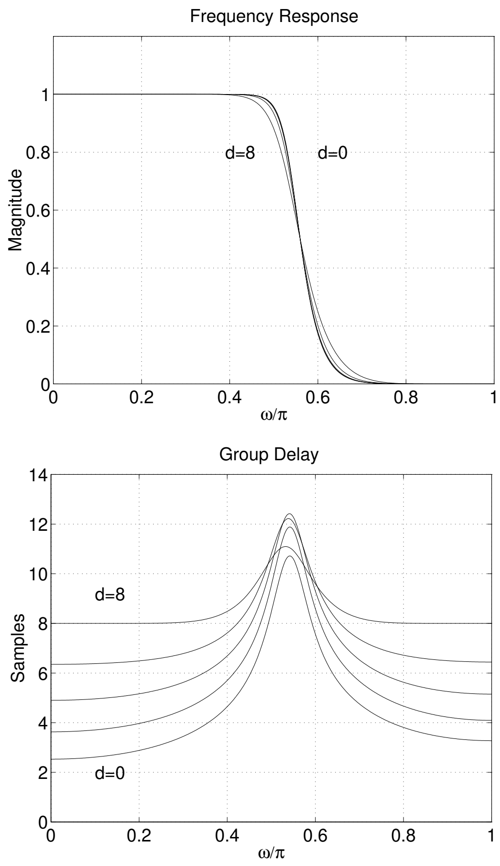


Figure 11: Sum of two allpass filters, with prescribed cut-off frequency $\omega_c = 0.56\pi$. $K = 4$, $L = 4$, $N = 9$, $d$ is varied from 0 to 8 in increments of 2. (It follows that $\tau$ is varied from $-4.5$ to $-0.5$ in increments of 1.)

$\omega = \pi$ respectively. $(ii)$ The application of this result to the design of lowpass filters realizable as allpass sums, whose magnitude response is flat at $\omega = 0$ and $\omega = \pi$ accordingly. This new class of maximally flat lowpass recursive digital filters can be made to have approximately linear phase in the passband (for $d = N - 1$), and the trade-off between delay and phase-linearity can be controlled by appropriate selection of the delay parameter $d$. In addition, the continuous variation of the cut-off frequency was described, in which case, for special values of $K$, $L$ and $d$ ($K = L$, $d = 0$), the classical Butterworth digital filter of odd degree is retrieved. Note that, with the classical Butterworth filter of degree $N$, it is not possible to adjust the delay (or phase-linearity). However, with the more general class of filters described in this paper, the adjustment of the delay becomes possible, and the trade-off between the delay and the phase-linearity can be chosen. Matlab programs for the the construction of the lowpass filters described in this paper, and the new maximally flat delay allpole filter on which it depends, are also provided.


## 6 Acknowledgments

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## A The Zeros of $D(z)$

As noted in Section 3, for the design of lowpass filters realizable as allpass sums, $\tau$ will be of the form $(d-N)/2$. It is therefore useful to examine the properties of $D(z)$ when $\tau$ is an integer multiple of one half. Of particular interest is the location of the zeros of $D(z)$ with respect to the unit circle $|z| = 1$. While poles of $1/D(z)$ outside the unit circle do not pose a problem (recall the factorization as in (9)), zeros *on* the unit circle will not give rise to useful solutions. Because $d \geq 0$, it is not necessary to consider integer multiples of one half that are more negative than $-N/2$.

It was found empirically that for $\tau \geq 0$, all the zeros of $D(z)$ lie inside the unit circle. The remaining values of $\tau$ to be considered are $-i/2$ for $i = 1, \ldots, N$. Let $q = 2\min(K, L) + 1$. Three ranges of $i$ will be considered: $1 \leq i \leq q - 1$, $i = q$, and $q + 1 \leq i \leq N$. The following observations were made, based on numerical observation.

1. $\tau = -i/2$ with $1 \leq i \leq q - 1$. For these values of $\tau$, $2\lfloor (i+1)/4 \rfloor$ zeros of $D(z)$ lie outside the unit circle. In addition, if $i = 2 \bmod 4$, then 2 zeros lie on the unit circle.

2. $\tau = -q/2$. For this value of $\tau$, $D(z) = (1 + z^{-1})^q z^{-N+q}$.

3. $\tau = -i/2$ with $q + 1 \leq i \leq N$. For $i$ even, $i/2 - 1$ zeros of $D(z)$ lie outside the unit circle and 2 zeros

lie on the unit circle. For $i$ odd, $i-q$ zeros of $D(z)$ lie outside the unit circle and $i+q-1$ zeros lie at $z=-1$.

Note that, of the negative values of $\tau$ considered, only those solutions, for which $1 \leq i \leq q-1$ and $i \neq 2 \bmod 4$, are free from zeros on the unit circle.

# B Restrictions

As noted in Section 3, some restrictions apply to the selection of $d$, $K$ and $L$. Although, as stated above, the group delay of $A(z)$ in (10) should approximate $d$ in both the passband and the stopband, this alone is not sufficient to ensure that the allpass sum (10) is lowpass. It is also necessary that $A(z) \approx -z^{-d}$ near $z=-1$. That means that for the allpass sum (10), $A(-1)$ should be $-(-1)^d$. On the other hand, (11) immediately yields $A(-1) = (-1)^N$. It follows that $d$ must have the same parity as $N+1$. Note that, in this case, $d-N$ is odd; therefore, the relevant values of $\tau$, namely $(d-N)/2$, are of the form $i/2$ where $i$ is an odd integer.

Two more conditions, in addition to the parity condition, are necessary. Recall that in Appendix A it was noted that, for $\tau = -i/2$ with $q \leq i \leq N$, the solution (5) possesses zeros on the unit circle. Since this precludes the utility of (5), we need only consider $i \leq q-1$. It follows that $d$ must satisfy the inequality $N-d \leq q-1$; that is $K+L-2\min(K,L) \leq d$, or equivalently, $|K-L| \leq d$. In combination with the parity condition, the condition $|K-L|+1 \leq d$ follows.

The last restriction is $d \leq N+1$. For values of $d$ greater than this, the unwrapped phase of $A(z)$ differs from the unwrapped phase of $z^{-d}$ by $\pi$ in $(0,\pi)$. Consequently, the magnitude response of the allpass sum will posses zeros in $(0,\pi)$, and points of unity gain between these zeros and $\omega = \pi$. Therefore, the magnitude response when $d > N+1$ is hardly lowpass.

Taken together, these considerations yield the requirements, ($i$) $|K-L|+1 \leq d \leq K+L+1$, and ($ii$) $d$ must have the same parity as $K+L+1$.

# C Programs

## C.1 Calculating the Flat Delay Filter

Although (5) appears rather cumbersome, the coefficients $a_n$ are very simple to compute because a recursive relationship exists. Let us begin with (6). Note that in (6)

$$\frac{a_{n+1}}{a_n} = -\frac{\binom{N}{n+1}}{\binom{N}{n}} \cdot \frac{(2\tau)_{n+1}(2\tau+N+1)_n}{(2\tau)_n(2\tau+N+1)_{n+1}} \tag{34}$$

$$= -\frac{N-n}{n+1} \cdot \frac{2\tau+n}{2\tau+N+1+n}. \tag{35}$$

With $a_0 = 1$, and the ratio (35), $a_{n+1}$ can be computed from $a_n$ by simply multiplying $a_n$ by this ratio. The Matlab command for the cumulative product, **cumprod**, makes it possible to write (6) in two lines:

```
n = 0:N-1;
a = cumprod([1, (n-N).*(2*t+n)./(n+1)./(2*t+N+1+n)]);
```

The procedure for computing $a_n$ in (5) is similar. Write $a_n$ as $a_n = \sum_{i=0}^{L} c_{i,n}$ where

$$c_{i,n} = \frac{(-4)^i(-1)^n(\tau)_i(n-i+1)_i(2\tau+2i)_{n-i}}{(2\tau+K+L+1)_n(K+L+1-i)_i}\binom{K+L}{n}\binom{L}{i}. \tag{36}$$

Then

$$\frac{c_{i,n+1}}{c_{i,n}} = -\frac{K+L-n}{n-i+1}\cdot\frac{2\tau+n+i}{2\tau+K+L+1+n}. \tag{37}$$

Therefore, $c_{i,n+1}$ can be computed by multiplying $c_{i,n}$ by this ratio. Note, however, that $c_{i,n}$ is zero for $n < i$, so the cumulative product operation must begin with the first nonzero term $c_{i,i}$. It turns out that these terms can also be computed by the cumulative product operation — the ratio $\frac{c_{i+1,i+1}}{c_{i,i}}$ is also a simple rational function. The simple Matlab program in Table 2 is obtained by putting these ratios together. The variable **c** denotes the $c_{i,i}$ term, and it is updated as **i** is incremented.

## C.2 Calculating the Allpass Sum

The program **mfaps**, for 'maximally flat allpass sum,' in Table 3, implements the design procedure described in this paper. Given the flatness parameters $K$ and $L$, and the delay $d$ in (8), the program returns the allpass filters $A_1(z)$ and $A_2(z)$, so that the allpass sum (8) is lowpass. The sum of the degrees of $A_1(z)$ and $A_2(z)$ is $K+L$. The frequency response magnitude has a flat characteristic at $\omega = 0$ and $\omega = \pi$. The allpass filters are represented by their denominators **a1** and **a2**. The program also gives **p** and **q** which represent the total lowpass response.

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Table 2: Matlab program for the flat delay filter.

```
function [b,a] = flatdelay(K,L,t)
% K, L : number of conditions at w=0, w=pi
% t : group delay
% b/a : digital allpole filter of degree K+L
% [b,a] = flatdelay(6,4,3.2); % example

N = K+L;
a = zeros(1,N+1);
c = 1;
for i = 0:L
   n = i:N-1;
   v = (n-N) ./ (n-i+1) .* (2*t+n+i) ./ (2*t+N+1+n);
   a = a + [zeros(1,i), cumprod([c, v])];
   c = c * 4 * (t+i) * (L-i) / (2*t+N+1+i) / (i+1);
end
b = sum(a);
```


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Table 3: Matlab program for the design of a lowpass filter realizable as an allpass sum.

```
function [a1,a2,p,q] = mfaps(K,L,d)
% Design of a maximally flat lowpass filter H(z) as the
% sum of two allpass filters: H(z) = z^(-d) A2(z) + A1(z).
% K, L : number of conditions at w=0, w=pi
% Note: two conditions must be satisfied
%   (1) abs(K-L)+1 <= d <= K+L+1
%   (2) d must be same parity as K+L+1
% a1, a2 : the denominators of the allpass filters A1(z), A2(z)
% p/q : overall transfer function

% check input for validity:
b1 = (abs(K-L)+1 <= d) & (d <= K+L+1);
b2 = rem(K+L+1-d,2)==0;
if ~(b1 & b2)
   disp('For this K and L, d must be one of the following:');
   disp((abs(K-L)+1):2:(K+L+1));
   break
end

[tmp,a] = flatdelay(K,L,(d-K-L)/2);
rts = roots(a);
v = abs(rts)<1;
a1 = real(poly(rts(v)));        % roots inside unit circle
a2 = real(poly(1./rts(~v)));    % roots outside unit circle

% compute overall transfer function p/q
p = [zeros(1,d), a] + [a(K+L+1:-1:1) zeros(1,d)];
q = conv(a1,a2);
p = p*sum(q)/sum(p);            % normalize
```

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