Symmetric Wavelet Tight Frames with Two Generators

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Abstract

This paper uses the UEP approach for the construction of wavelet tight frames with two (anti-) symmetric wavelets, and provides some results and examples that complement recent results by Q. Jiang. A description of a family of solutions when the lowpass scaling filter is of even-length is provided. When one wavelet is symmetric and the other is anti-symmetric, the wavelet filters can be obtained by a simple procedure based on matching the roots of associated polynomials. The design examples in this paper begin with the construction of a lowpass filter $h_0(n)$ that is designed so as to ensure that both wavelets have at least a specified number of vanishing moments.

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1 Introduction

The development of wavelet frames is motivated in part by the performance gains provided by some *expansive* wavelet transforms in comparison with critically-sampled transforms. An expansive transform expands an N-sample data vector to M coefficients with $M > N$.

There are several ways to implement an expansive discrete wavelet transform (DWT). For example, an undecimated DWT can be implemented by removing the down-sampling operations in the usual DWT implementation [4]. The undecimated DWT is a shift-invariant form of the DWT and avoids some of the artifacts that arise when the critically-sampled DWT is used for signal/image denoising and image enhancement. The undecimated DWT is expansive by a factor of $(J+1)$ where J is the number of stages (scales) in the implementation. Another example of an expansive wavelet transform is the dual-tree complex wavelet transform ($\mathbb{C}$WT) which is implemented by performing two DWTs in parallel on the same data [12]. The $\mathbb{C}$WT is designed to behave as an analytic wavelet transform which gives it several attractive properties in signal processing applications. The $\mathbb{C}$WT is nearly shift-invariant and is expansive by a factor of two independent of the number of stages implemented. The undecimated DWT and the $\mathbb{C}$WT both provide significant performance gains in some signal processing applications[1].

Both the undecimated DWT and the $\mathbb{C}$WT are implemented using two-channel digital filter banks having the perfect reconstruction property. As such, the number of free variables in the design of wavelets for the undecimated DWT and $\mathbb{C}$WT is no greater than that for the critically-sampled DWT. However, an expansive wavelet transform implemented using an oversampled filter bank can have more degrees of freedom for wavelet design, which offers more flexibility in attaining desired properties (wavelet smoothness, symmetry, etc). An expansive DWT implemented using an oversampled filter bank gives wavelet coefficients corresponding to the representation of a signal in a wavelet frame. Examples of wavelet frames for which the implementation requires only finite impulse response (FIR) filters have been described in [1, 2, 7, 11, 13, 14, 16, 17, 18, 19].

Some of the first wavelet tight frames that can be implemented using a bank of FIR filters were described in [17]. Given the compactly supported scaling function $\phi(t)$ (equivalently, a lowpass filter $h_0(n)$), it is described in [2] how to obtain two compactly supported wavelets so that the

[1] Software for the $\mathbb{C}$WT and denoising examples are available at `http://taco.poly.edu/WaveletSoftware/`

dyadic dilations and translations of the wavelets form a tight frame for $L^2(\mathbb{R})$. In addition, when $\phi(t)$ is symmetric, it is shown how to obtain three (anti-) symmetric wavelets generating a tight frame. Alternative methods for constructing the wavelets from the scaling function are described in [13]. Given a compactly supported symmetric scaling function $\phi(t)$, it is generally impossible to obtain a wavelet tight frame with only two (anti-) symmetric wavelets. However, [14] provides a condition that the lowpass filter $h_0(n)$ must satisfy so that this becomes possible. These papers provide examples of wavelet frames where the wavelets are (anti-) symmetric; however, in many of the examples one of the wavelets in each tight frame has only one vanishing moment. In [18, 19] wavelet tight frames are developed where each wavelet has more than one vanishing moment but none are (anti-) symmetric. A recent paper [11] presents a complete factorization of FIR filters for tight frames with two (anti-) symmetric wavelets and provides examples where each wavelet has more than one vanishing moment. In addition, the possible length combinations and associated symmetries are derived in [11].

The construction of the wavelet frames in these papers follows the unitary extension principle (UEP) — the completion of a paraunitary matrix given a single row or column. Recently, a new approach to the construction of tight wavelet frames with compactly supported wavelets has been introduced in [3, 8]. In these papers, for the same scaling function $\phi(t)$ the wavelets can have more vanishing moments than was previously possible. This development uses the notion of a *vanishing-moment recovery* (VMR) function ($S(z)$ in [3], $\Theta(\omega)$ in [8]). Examples of tight wavelet frames with short support, symmetry, and good vanishing moment properties are given in [3, 8]. Although the scaling function and wavelets are themselves of compact support, the implementation of the transform for discrete data requires prefiltering (and/or postfiltering) the discrete data with an infinite impulse response (IIR) filter — this is a convolution of the discrete data with a non-compactly supported sequence. However, the remaining filter-bank-based implementation requires only FIR filters.

In this paper, we revisit the UEP approach to the construction of wavelet tight frames with two (anti-) symmetric wavelets and provide some results and examples that complement those in [11]. We provide a description of a family of solutions when the lowpass scaling filter $h_0(n)$ is of even-length. In addition, we construct symmetric scaling functions $\phi(t)$ for which wavelet tight frames with two (anti-) symmetric wavelets exist, and describe how two (anti-) symmetric wavelets can be obtained by matching the roots of associated polynomials. The examples of wavelet frames below can also be obtained using the approach in [11] where parameters in the factorization are determined. The design examples in this paper begin with the construction of a lowpass filter $h_0(n)$ that is designed so as to ensure that both the wavelets have at least a specified number of vanishing moments.

2 Preliminaries

As in [2, 13, 17, 18], following the multiresolution framework, the scaling function and wavelets are defined through the dilation and wavelet equations:

$$\phi(t) = \sqrt{2} \sum_n h_0(n)\, \phi(2\,t - n) \tag{1}$$

$$\psi_i(t) = \sqrt{2} \sum_n h_i(n)\, \phi(2\,t - n), \qquad i = 1, 2 \tag{2}$$

where $h_i(n)$, $n \in \mathbf{Z}$, are the filters of a three-channel filter bank. Each branch of the filter bank is sub-sampled by two; see [18]. In this paper, we consider only real-valued $h_i(n)$ of compact support (FIR). If $h_i(n)$ satisfy the perfect reconstruction conditions given below and if $\phi(t)$ is sufficiently regular, then the dyadic dilations and translations of $\psi_i(t)$ form a tight frame for $L^2(\mathbb{R})$.

In this paper, the Z-transform of $h(n)$ is given by $H(z) = \sum_n h(n)\, z^{-n}$. The discrete-time Fourier transform of $h(n)$ is given by $H(e^{j\,\omega})$.

2.1 Perfect Reconstruction Conditions

In the implementation, we are interested in the case where the synthesis filter bank (inverse transform) constitutes the transpose of the analysis filter bank (forward transform). This requires that the synthesis filters be the time-reversed versions of analysis filters. Using multirate system theory, the perfect reconstruction (PR) conditions for the three-channel filter bank are the following two equations.

$$H_0(z)\, H_0(1/z) + H_1(z)\, H_1(1/z) + H_2(z)\, H_2(1/z) = 2 \tag{3}$$

$$H_0(-z)\, H_0(1/z) + H_1(-z)\, H_1(1/z) + H_2(-z)\, H_2(1/z) = 0 \tag{4}$$

The polyphase components are defined so that:

$$H_i(z) = H_{i0}(z^2) + z^{-1}\, H_{i1}(z^2) \quad \text{for } i = 0, 1, 2.$$

The perfect reconstruction conditions can also be written in matrix form as:

$$H^t(1/z)\, H(z) = I$$

where

$$H(z) = \begin{bmatrix} H_{00}(z) & H_{01}(z) \\ H_{10}(z) & H_{11}(z) \\ H_{20}(z) & H_{21}(z) \end{bmatrix}$$

is the *polyphase* matrix. Equivalently,

$$\begin{bmatrix} H_{10}(1/z) & H_{20}(1/z) \\ H_{11}(1/z) & H_{21}(1/z) \end{bmatrix} \begin{bmatrix} H_{10}(z) & H_{11}(z) \\ H_{20}(z) & H_{21}(z) \end{bmatrix} = \begin{bmatrix} 1 - H_{00}(z)\, H_{00}(1/z) & -H_{00}(1/z)\, H_{01}(z) \\ -H_{00}(z)\, H_{01}(1/z) & 1 - H_{01}(z)\, H_{01}(1/z) \end{bmatrix} \tag{5}$$

This form of the PR conditions has been investigated in detail and used in [3, 7, 13] for a more general form of the problem.

From [2], we have a requirement on $h_0(n)$: If $h_0(n)$ is compactly supported then a solution $(h_1(n),\, h_2(n))$ to (3,4) exists if and only if

$$|H_0(e^{j\,\omega})|^2 + |H_0(e^{j\,(\omega+\pi)})|^2 \leq 2. \tag{6}$$

From [14], we have the result: If $h_0(n)$ is symmetric, compactly supported, and satisfies (6) then an (anti-) symmetric solution $(h_1(n),\, h_2(n))$ to (3,4) exists if and only if all the roots of

$$2 - H_0(z)\, H_0(1/z) - H_0(-z)\, H_0(-1/z) \tag{7}$$

have even multiplicity.

2.2 About Symmetries

The goal is to design a set of three filters satisfying the perfect reconstruction conditions where the low-pass filter $h_0(n)$ is symmetric and the filters $h_1(n)$ and $h_2(n)$ are each either symmetric or anti-symmetric. There are two cases. Case I will denote the case where $h_1(n)$ is symmetric and $h_2(n)$ is anti-symmetric. Case II will denote the case where both $h_1(n)$ and $h_2(n)$ are anti-symmetric.

The symmetry condition for $h_0(n)$ is:

$$h_0(n) = h_0(N-1-n). \tag{8}$$

In this paper, we deal exclusively with the case of even-length filters. When N is even, it follows from the symmetry condition (8) that the polyphase components of $H_0(z)$ are related as

$$H_{01}(z) = z^{-(N/2-1)}\, H_{00}(1/z). \tag{9}$$

We will also need to see what the symmetry conditions imply about the polyphase components of $H_1(z)$ and $H_2(z)$. We will examine Case I and Case II separately.

3 Case I

Solutions for Case I can be obtained from solutions where $h_2(n)$ is a time-reversed version of $h_1(n)$ (and where neither filter is (anti-) symmetric). To show this, suppose that $\{h_0, h_1, h_2\}$ satisfy the PR conditions and that

$$h_2(n) = h_1(N-1-n). \tag{10}$$

Then, by defining

$$h_1^{\text{new}}(n) = \frac{1}{\sqrt{2}}\,(h_1(n) + h_2(n-2\,d)) \tag{11}$$

$$h_2^{\text{new}}(n) = \frac{1}{\sqrt{2}}\,(h_1(n) - h_2(n-2\,d)) \tag{12}$$

with $d \in \mathbf{Z}$, the filters $\{h_0, h_1^{\text{new}}, h_2^{\text{new}}\}$ also satisfy the PR conditions, and h_1^{new}, h_2^{new}, are symmetric and anti-symmetric:

$$h_1^{\text{new}}(n) = h_1^{\text{new}}(N_2-1-n) \tag{13}$$

$$h_2^{\text{new}}(n) = -h_2^{\text{new}}(N_2-1-n) \tag{14}$$

where $N_2 = N + 2\,d$. Therefore, we can focus on developing solutions having the properties (8) and (10).

The condition (10) requires that the polyphase components be related as

$$H_{20}(z) = z^{-(N/2-1)}\, H_{11}(1/z) \tag{15}$$

$$H_{21}(z) = z^{-(N/2-1)}\, H_{10}(1/z). \tag{16}$$

Using (9), (15) and (16), the polyphase form of the PR conditions (5) can be written as

$$\begin{bmatrix} H_{10}(1/z) & z^M H_{11}(z) \\ H_{11}(1/z) & z^M H_{10}(z) \end{bmatrix} \begin{bmatrix} H_{10}(z) & H_{11}(z) \\ z^{-M} H_{11}(1/z) & z^{-M} H_{10}(1/z) \end{bmatrix} = \begin{bmatrix} 1 - H_{00}(z) H_{00}(1/z) & -z^{-M} H_{00}^2(1/z) \\ -z^M H_{00}^2(z) & 1 - H_{00}(z) H_{00}(1/z) \end{bmatrix} \tag{17}$$

where $M := N/2 - 1$. Taking the determinant of both sides and simplifying, we obtain

$$U(1/z)\, U(z) = 1 - 2\, H_{00}(z)\, H_{00}(1/z) \tag{18}$$

where

$$U(z) = H_{10}(z)\, H_{10}(1/z) - H_{11}(z)\, H_{11}(1/z). \tag{19}$$

Note that (18) also gives

$$|H_{00}(e^{j\,\omega})|^2 \leq 0.5.$$

Note that $U(z) = U(1/z)$ so we obtain

$$U^2(z) = 1 - 2\, H_{00}(z)\, H_{00}(1/z). \tag{20}$$

Therefore, all the zeros of $1 - 2\, H_{00}(z)\, H_{00}(1/z)$ must be of even degree. This is a form of the condition from [14] stating that (7) must have all its roots of even degree.

Multiplying (17) out we get two distinct equations:

$$H_{10}(z)\, H_{10}(1/z) + H_{11}(z)\, H_{11}(1/z) = 1 - H_{00}(z)\, H_{00}(1/z) \tag{21}$$

$$2\, H_{11}(z)\, H_{10}(1/z) = -z^{-M}\, H_{00}^2(1/z). \tag{22}$$

For Case I, the PR conditions (5) are equivalent to equations (21) and (22).

Lemma 1 The filters $\{h_0, h_1, h_2\}$ with symmetries (8) and (10) satisfy the perfect reconstruction conditions if their polyphase components are given by

$$H_{00}(z) = z^{-M/2} \sqrt{2}\, A(z)\, B(1/z) \tag{23}$$

$$H_{10}(z) = A^2(z) \tag{24}$$

$$H_{11}(z) = -B^2(z) \tag{25}$$

where $A(z)$ and $B(z)$ satisfy

$$A(z)\, A(1/z) + B(z)\, B(1/z) = 1. \tag{26}$$

In this system the filter $h_0(n)$ is symmetric [as in (8)] and $h_2(n)$ is a time-reversed version of $h_1(n)$ [as in (10)]. To prove the lemma, we need to verify that PR conditions (21) and (22) for Case I are satisfied. Substituting (23)-(25) into (21) we obtain

$$A^2(z)\,A^2(1/z) + B^2(z)\,B^2(1/z) = 1 - 2\,A(z)\,A(1/z)\,B(z)\,B(1/z),$$

or

$$A^2(z)\,A^2(1/z) + 2\,A(z)\,A(1/z)\,B(z)\,B(1/z) + B^2(z)\,B^2(1/z) = 1,$$

or

$$(A(z)\,A(1/z) + B(z)\,B(1/z))^2 = 1.$$

Using (26) gives $1 = 1$, verifying that condition (21) is satisfied. Substituting (23)-(25) into (22) immediately verifies that condition (22) is also satisfied.

To show why the polyphase components have roots of even degree, note that from (22) it follows that if $H_{10}(1/z)$ and $H_{11}(z)$ have no common roots then they each have roots of even degree. But $H_{11}(z)$ and $H_{10}(1/z)$ can not have any common roots: Suppose $H_{11}(z_o) = H_{10}(1/z_o) = 0$, then from (22) $H_{00}(1/z_o) = 0$. Equation (21) then gives $0 = 1$. Therefore, $H_{11}(z)$ and $H_{10}(1/z)$ can not have any common roots and they each have roots of even degree.

Remarks

1.) For $\phi(t)$ in (1) to exist and to be well-defined, it is necessary that $H_0(1) = \sqrt{2}$. From (23) this requires that $A(1)\,B(1) = 0.5$. From the PR condition (3) it follows in addition that $H_1(1) = 0$, and hence that $A^2(1) - B^2(1) = 0$. Therefore, $A(z)$ and $B(z)$ can be normalized so that

$$A(1) = B(1) = \frac{1}{\sqrt{2}}. \tag{27}$$

2.) Consider the problem: Given $H_0(z)$ satisfying (20), what is the simplest way to find $H_1(z)$ and $H_2(z)$? That is, what is the simplest way to find $A(z)$ and $B(z)$ (assuming a solution of this form exists)?

Note that $U(z)$ can be found from (20) by factorization. Substituting (23)-(25) into (19) and using (26) gives

$$
\begin{aligned}
U(z) &= A^2(z)\,A^2(1/z) - B^2(z)\,B^2(1/z) && (28)\\
&= [A(z)\,A(1/z) + B(z)\,B(1/z)]\,[A(z)\,A(1/z) - B(z)\,B(1/z)] && (29)\\
&= A(z)\,A(1/z) - B(z)\,B(1/z) && (30)\\
&= 2\,A(z)\,A(1/z) - 1 && (31)\\
&= 1 - 2\,B(z)\,B(1/z) && (32)
\end{aligned}
$$

so

$$A(z)\,A(1/z) = 0.5 + 0.5\,U(z) \tag{33}$$

and

$$B(z)\,B(1/z) = 0.5 - 0.5\,U(z). \tag{34}$$

Therefore, $A(z)$ is a factor of both $H_{00}(z)$ and $0.5 + 0.5\,U(z)$, so $A(z)$ can be determined by identifying the common roots. Similarly, $B(z)$ is a factor of both $H_{00}(1/z)$ and $0.5 - 0.5\,U(z)$, so $B(z)$ can likewise be determined. This is illustrated in Example 1 below.

3.) What behavior for $A(z)$ and $B(z)$ is required so that (i) $H_0(z) \approx 0$ around $z = -1$ and (ii) $H_1(z) \approx 0$ around $z = 1$? These conditions are important for applications of wavelet transforms.

Let's examine $H_1(z)$ first. The filter $H_1(z)$ is given by

$$
\begin{aligned}
H_1(z) &= H_{10}(z^2) + z^{-1}\,H_{11}(z^2) && (35)\\
&= A^2(z^2) - z^{-1}\,B^2(z^2). && (36)
\end{aligned}
$$

The condition $H_1(e^{j\,\omega}) \approx 0$ around $\omega = 0$ requires that

$$A^2(e^{j\,2\,\omega}) - e^{-j\omega}\,B^2(e^{j\,2\,\omega}) \approx 0 \quad \text{around } \omega = 0$$

or

$$
\begin{aligned}
A^2(e^{j\,\omega}) &\approx e^{-j\,\omega/2}\,B^2(e^{j\,\omega}) && (37)\\
A(e^{j\,\omega}) &\approx e^{-j\,\omega/4}\,B(e^{j\,\omega}) && (38)
\end{aligned}
$$

around $\omega = 0$. Using properties of the discrete-time Fourier transform, this can be written informally as an approximate quarter-sample delay

$$a(n) \approx b(n - 0.25)$$

for low frequencies. As $h_2(n)$ is a time-reversed version of $h_1(n)$, $H_2(e^{j\,\omega})$ will approximate zero around $\omega = 0$ equally well as $H_1(e^{j\,\omega})$.

Now let's examine $H_0(z)$. $H_0(z)$ is given by

$$H_0(z) = H_{00}(z^2) + z^{-1}\, H_{01}(z^2) \tag{39}$$

$$= \sqrt{2}\, z^{-M}\, [A(z^2)\, B(1/z^2) + z^{-1}\, A(1/z^2)\, B(z^2)]. \tag{40}$$

The condition $H_0(e^{j\,\omega}) \approx 0$ around $\omega = \pi$ requires that

$$A(e^{j\,2\,\omega})\, B(e^{-j\,2\,\omega}) \approx -e^{-j\,\omega}\, A(e^{-j\,2\,\omega})\, B(e^{j\,2\,\omega}) \qquad \text{around } \omega = \pi \tag{41}$$

$$\Rightarrow \qquad A(e^{j\,\omega})\, B(e^{-j\,\omega}) \approx -e^{-j\,\omega/2}\, A(e^{-j\,\omega})\, B(e^{j\,\omega}) \qquad \text{around } \omega = 2\,\pi \tag{42}$$

$$\Rightarrow \qquad A(e^{j\,\omega})\, B(e^{-j\,\omega}) \approx e^{-j\,\omega/2}\, A(e^{-j\,\omega})\, B(e^{j\,\omega}) \qquad \text{around } \omega = 0 \tag{43}$$

$$\Rightarrow \qquad A(e^{j\,\omega})\, \overline{B(e^{j\,\omega})} \approx e^{-j\,\omega/2}\, \overline{A(e^{j\,\omega})}\, B(e^{j\,\omega}) \qquad \text{around } \omega = 0 \tag{44}$$

Taking the angle of both sides gives

$$\angle A(e^{j\,\omega}) - \angle B(e^{j\,\omega}) \approx -\omega/2 - \angle A(e^{j\,\omega}) + \angle B(e^{j\,\omega}) \qquad \text{around } \omega = 0$$

or

$$\angle A(e^{j\,\omega}) \approx -\omega/4 + \angle B(e^{j\,\omega}) \qquad \text{around } \omega = 0. \tag{45}$$

Note that (38) already implies this.

3.1 Example 1

This example uses the condition (20) from [14] and lemma 1 to design a (anti-) symmetric wavelet tight frame. We will first obtain a minimal-length low-pass filter $h_0(n)$ satisfying (20). We will then obtain $A(z)$ and $B(z)$ by factorization and root selection. Lastly, (anti-) symmetric filters $h_1(n)$ and $h_2(n)$ will be obtained.

Our construction of $h_0(n)$ will be based on the maximally-flat lowpass even-length FIR filter [5, 10], which has the following transfer function:

$$F^{(M,L)}(z) = \frac{1}{2}\,(1+z^{-1}) \left(\frac{z+2+z^{-1}}{4}\right)^M \sum_{n=0}^{L} \binom{M+n-0.5}{n} \left(\frac{-z+2-z^{-1}}{4}\right)^n. \tag{46}$$

To evaluate the binomial coefficient for fractional values of the upper entry, we can use the Gamma function Γ. The Gamma function interpolates the factorial function on the integers, $\Gamma(n+1) = n!$ for $n \in \mathbf{Z}$, so we have

$$\binom{M+n-0.5}{n} = \frac{\Gamma(M+0.5+n)}{\Gamma(M+0.5)\,\Gamma(n+1)}.$$

The normalization in (46) is such that $F^{(M,L)}(1) = 1$.

If $\sqrt{2}\,F^{(M,L)}(z)$ is used as a scaling filter $H_0(z)$ then each wavelet will have at least $L+1$ vanishing moments[2]. Unfortunately, setting $H_0(z) := F^{(M,L)}(z)$ gives an $H_0(z)$ that does not satisfy (20). [That is, $1 - 2\,H_{00}(z)\,H_{00}(1/z)$ will not have roots of even degree[3].] However, by using a linear combination of various $F^{(m,l)}(z)$, we can obtain a filter $H_0(z)$ that does satisfy (20). For example, if we set

$$H_0(z) := z^{-4}\,\sqrt{2}\,\left(\alpha\,F^{(2,1)}(z) + (1-\alpha)\,F^{(3,1)}(z)\right), \tag{47}$$

then for special values of α, $H_0(z)$ satisfies (20). In addition, $H_0(z)$ will have $(1+z^{-1})^5$ as a factor and each wavelet will have at least two vanishing moments. The term z^{-4} in (47) makes $H_0(z)$ causal [$h_0(n) = 0$ for $n < 0$]. From (46) we have

$$F^{(2,1)}(z) = \frac{z^2}{2^8}\,(1+z^{-1})^5\,(-5\,z + 18 - 5\,z^{-1})$$

$$F^{(3,1)}(z) = \frac{z^3}{2^{10}}\,(1+z^{-1})^7\,(-7\,z + 22 - 7\,z^{-1}).$$

For convenience, we can write $H_{00}(z)$ as

$$H_{00}(z) = \alpha\,G_1(z) + (1-\alpha)\,G_0(z)$$

with

$$G_0(z^2) := \frac{z^{-4}}{\sqrt{2}}\left[F^{(3,1)}(z) + F^{(3,1)}(-z)\right] \tag{48}$$

$$G_1(z^2) := \frac{z^{-4}}{\sqrt{2}}\left[F^{(2,1)}(z) + F^{(2,1)}(-z)\right] \tag{49}$$

[2]This is because $1 - F^{(M,L)}(z)\,F^{(M,L)}(1/z)$ has $(1-z)^{2L+2}$ as a factor.

[3]The few exceptions, identified in [14], have $L = 0$ so at least one wavelet will have only one vanishing moment.

which is derived using

$$H_{00}(z^2) = 0.5\, H_0(z) + 0.5\, H_0(-z).$$

We can find values of α for which $H_0(z)$ in (47) satisfies (20) as follows. We can write

$$\begin{aligned} 1 - 2\, H_{00}(z)\, H_{00}(1/z) = {} & [1 - 2\, G_0(z)\, G_0(1/z)] + \\ & \alpha\, [4\, G_0(z)\, G_0(1/z) - 2\, G_0(z)\, G_1(1/z) - 2\, G_0(1/z)\, G_1(z)] + \\ & \alpha^2\, [2\, G_0(z)\, G_1(1/z) + 2\, G_0(1/z)\, G_1(z) - 2\, G_0(z)\, G_0(1/z) - 2\, G_1(z)\, G_1(1/z)]. \end{aligned} \tag{50}$$

The right-hand side of (50) can be simplified using the change of variables used in [6, 10],

$$x = \frac{-z + 2 - z^{-1}}{4}.$$

The right-hand side of (50) becomes

$$\frac{x^2}{2^{10}} \left[(1008 + 84\, x - 189\, x^2) + \alpha\, (-448 + 56\, x + 238\, x^2) - \alpha^2\, 49\, x^2\right],$$

or

$$x^2 \left(P_0(\alpha) + P_1(\alpha)\, x + P_2(\alpha)\, x^2\right). \tag{51}$$

The required values of α are those for which the polynomial in (50) has roots of even degree. The term x^2 in (51) implies that $1 - 2\, H_{00}(z)\, H_{00}(1/z)$ has a root at $z = 1$ of multiplicity 4. The remaining roots of (51) must be of even degree. As (51) is the product of x^2 and a quadratic polynomial in x, we can find α by setting the discriminant, $D(\alpha) = P_1^2(\alpha) - 4\, P_0(\alpha)\, P_2(\alpha)$, to zero. The discriminant is a third degree polynomial in α:

$$D(\alpha) = \frac{7^2}{2^{16}}\, (-112\, \alpha^3 + 800\, \alpha^2 - 1644\, \alpha + 981).$$

The roots of $D(\alpha)$ are approximately $\{1.0720,\ 2.0140,\ 4.0568\}$. Setting α to the smallest of the three roots produces the smoothest scaling function, illustrated in Figure 1. (The corresponding Sobolev exponents, computed using [9, 21], are 3.20, 2.59, and 1.78 respectively. The corresponding Hölder exponents are 2.78, 2.22, and 1.47 respectively. Because the filter $h_0(n)$ is symmetric and positive on the unit circle the Hölder exponents can be computed as described in [20].) The filter $h_0(n)$ corresponding to this value of α is listed in Table 1. We next find $h_1(n)$ and $h_2(n)$ corresponding to this value of α.

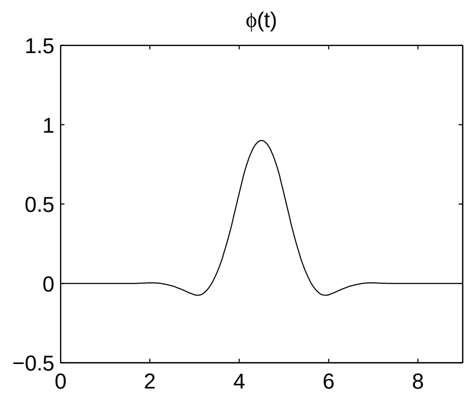

Figure 1: The symmetric scaling function, $\phi(t)$, of Example 1. $H_0(z)$ has $(1 + z^{-1})^5$ as a factor. The filter $h_0(n)$ is given in Table 1.

We can find $H_1(z)$ and $H_2(z)$ from $A(z)$ and $B(z)$. First, note that the roots of $H_{00}(z)$ are, approximately,

$$\{36.0738,\ 23.7888,\ -0.4225,\ 0.1067\}. \tag{52}$$

To find $A(z)$ and $B(z)$, we first find $U(z)$ from (20) by factorization. The sequence $u(n)$ will be symmetric because $U(z) = U(1/z)$. Next, we find the roots of $0.5 + 0.5\,U(z)$. They are approximately

$$\{23.7888,\ 9.3746,\ 0.1067,\ 0.0420\}. \tag{53}$$

The roots of $0.5 - 0.5\,U(z)$ are approximately

$$\{36.0738,\ -2.3668,\ -0.4225,\ 0.0277\}. \tag{54}$$

From (23) and (33), the roots of $A(z)$ are those roots common to (53) and (52), so the roots of $A(z)$ are $\{23.7888,\ 0.1067\}$. Similarly, from (23) and (34), the roots of $B(z)$ are those roots common to (54) and the reciprocals of (52). So the roots of $B(z)$ are $\{-2.3668,\ 0.0277\}$. Using the normalization (27) we obtain

$$A(z) = -0.0347 + 0.8300\,z^{-1} - 0.0881\,z^{-2} \tag{55}$$

$$B(z) = 0.2160 + 0.5053\,z^{-1} - 0.0142\,z^{-2} \tag{56}$$

It was shown above in (38) and (45) that the angle of $B(e^{j\,\omega})/A(e^{j\,\omega})$ should approximate $0.25\,\omega$ around $\omega = 0$. For $A(z)$ and $B(z)$ in (55) and (56), this approximation is illustrated in Figure 2.

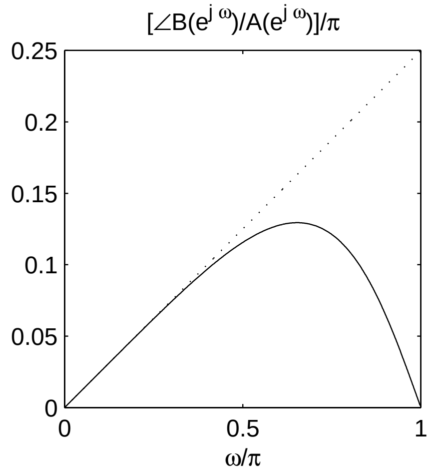

Figure 2: The solid line shows the angle of $B(e^{j\,\omega})/A(e^{j\,\omega})$. This confirms its approximation to $0.25\,\omega$ around $\omega = 0$ according to (38). The dotted line shows $0.25\,\omega/\pi$ for reference.

The wavelet filters, $h_1(n)$ and $h_2(n)$, given by (24) and (25), are tabulated in Table 1. They satisfy $h_2(n) = h_1(9-n)$ and are not symmetric. The corresponding wavelets, illustrated in Figure 3, satisfy $\psi_2(t) = \psi_1(9-t)$ and have two vanishing moments each. These two wavelets generate a wavelet tight frame.

To obtain a wavelet tight frame with (anti-) symmetric wavelets, we can use (11) and (12). Using $d = 1$, the resulting filters that are obtained are listed in Table 2. The two wavelets generated by these filters are illustrated in Figure 4. The wavelet $\psi_1(t)$ has two vanishing moments, while $\psi_2(t)$ has three vanishing moments. These two wavelets generate a wavelet tight frame. Other integer values of d also generate wavelet tight frames with (anti-) symmetric wavelets.

Note that although we found the polynomials $A(z)$ and $B(z)$ from $H_0(z)$ in Example 1, lemma 1 does not guarantee it in general. An alternative approach is to start with $A(z)$ and $B(z)$ given by some unknown parameters, then solve equations so that $H_0(z)$ has $(1 + z^{-1})^L$ as a factor and so that the wavelets have the desired number of vanishing moments.

Also note that by appropriately designing $H_0(z)$ it is guaranteed that the wavelets (however they are obtained) will have at least a specified number of vanishing moments. Specifically, for UEP, the minimum number of vanishing moments among the wavelets is determined by the number of vanishing derivatives of $|H(e^{j\omega})|$ at $\omega = 0$, see [18]. For this reason, in Example 1, we first obtained

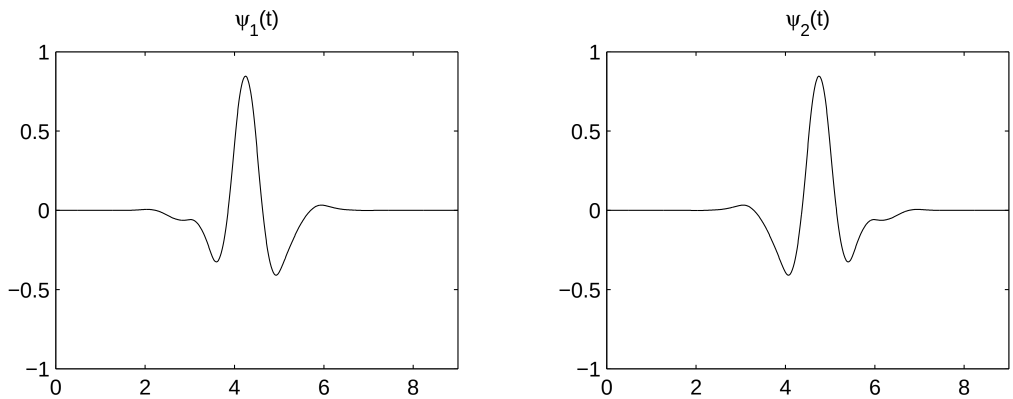

Figure 3: The two wavelets $\psi_1(t)$, $\psi_2(t)$ of Example 1 where $\psi_2(t) = \psi_1(9-t)$. Both wavelets have two vanishing moments. The filters $h_i(n)$ are given in Table 1.

the minimal-length symmetric solution for which $H_0(z)$ has $(1+z^{-1})^5$ as a factor and for which each wavelet has at least two vanishing moments.

4 Case II

Dyadic-wavelet tight frames with two anti-symmetric compactly supported wavelets can be obtained with filters $h_i(n)$ satisfying the following symmetry conditions:

$$h_0(n) = h_0(N-1-n) \tag{57}$$

$$h_1(n) = -h_1(N-1-n) \tag{58}$$

$$h_2(n) = -h_2(N-3-n). \tag{59}$$

These symmetry conditions lead to the following relationship between the polyphase components of each filter:

$$H_{01}(n) = z^{-M}\, H_{00}(1/z) \tag{60}$$

$$H_{11}(n) = -z^{-M}\, H_{10}(1/z) \tag{61}$$

$$H_{21}(n) = -z^{-(M-1)}\, H_{20}(1/z) \tag{62}$$

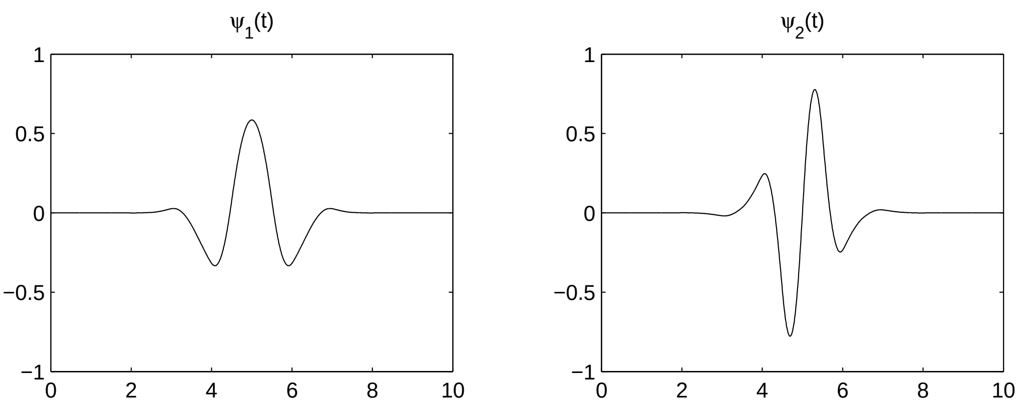

Figure 4: The (anti-) symmetric wavelets of Example 1. $\psi_1(t)$ has two vanishing moments. $\psi_2(t)$ has three vanishing moments. The filters $h_i(n)$ are given in Table 2.

Table 1: Coefficients for Example 1.

n	$h_0(n)$	$h_1(n)$	$h_2(n)$
0	0.00069616789827	0.00120643067872	-0.00020086099895
1	-0.02692519074183	-0.04666026144290	0.00776855801988
2	-0.04145457368921	-0.05765656504458	0.01432190717031
3	0.19056483888762	-0.21828637525088	-0.14630790303599
4	0.58422553883170	0.69498947938197	-0.24917440947758
5	0.58422553883170	-0.24917440947758	0.69498947938197
6	0.19056483888762	-0.14630790303599	-0.21828637525088
7	-0.04145457368921	0.01432190717031	-0.05765656504458
8	-0.02692519074183	0.00776855801988	-0.04666026144290
9	0.00069616789827	-0.00020086099895	0.00120643067872

Table 2: Coefficients for Example 1.

n	$h_0(n)$	$h_1(n)$	$h_2(n)$
0	0.00069616789827	-0.00014203017443	0.00014203017443
1	-0.02692519074183	0.00549320005590	-0.00549320005590
2	-0.04145457368920	0.01098019299363	-0.00927404236573
3	0.19056483888763	-0.13644909765612	0.07046152309968
4	0.58422553883167	-0.21696226276259	0.13542356651691
5	0.58422553883167	0.33707999754362	-0.64578354990472
6	0.19056483888763	0.33707999754362	0.64578354990472
7	-0.04145457368920	-0.21696226276259	-0.13542356651691
8	-0.02692519074183	-0.13644909765612	-0.07046152309968
9	0.00069616789827	0.01098019299363	0.00927404236573
10	0	0.00549320005590	0.00549320005590
11	0	-0.00014203017443	-0.00014203017443

where $M := N/2 - 1$. Using (60), (61), (62), the polyphase form of the PR conditions (5) can be written as

$$\begin{bmatrix} H_{10}(1/z) & H_{20}(1/z) \\ -z^{M} H_{10}(z) & -z^{M-1} H_{20}(z) \end{bmatrix} \begin{bmatrix} H_{10}(z) & -z^{-M} H_{10}(1/z) \\ H_{20}(z) & z^{-(M-1)} H_{20}(1/z) \end{bmatrix} = \begin{bmatrix} 1 - H_{00}(z) H_{00}(1/z) & -z^{-M} H_{00}^2(1/z) \\ -z^{M} H_{00}^2(z) & 1 - H_{00}(z) H_{00}(1/z) \end{bmatrix}. \tag{63}$$

Taking the determinant of both sides and simplifying, we obtain

$$U(1/z)\, U(z) = 1 - 2\, H_{00}(z)\, H_{00}(1/z)$$

where

$$U(z) = H_{10}(z)\, H_{20}(1/z) - z^{-1}\, H_{10}(1/z)\, H_{20}(z). \tag{64}$$

Note that $U(1/z) = -z^{-1}\, U(z)$ so we obtain

$$-z^{-1}\, U^2(z) = 1 - 2\, H_{00}(z)\, H_{00}(1/z). \tag{65}$$

Therefore all the zeros of $1 - 2\, H_{00}(z)\, H_{00}(1/z)$ must be of even degree. This is the form for Case II of the condition from [14] stating that (7) must have all its roots of even degree.

Multiplying (63) out we get two distinct equations:

$$H_{10}(z)\, H_{10}(1/z) + H_{20}(z)\, H_{20}(1/z) = 1 - H_{00}(z)\, H_{00}(1/z) \tag{66}$$

$$H_{10}^2(z) + z^{-1}\, H_{20}^2(z) = H_{00}^2(z). \tag{67}$$

For Case II, the PR conditions (5) are equivalent to equations (66) and (67). It can be verified directly that the following form for the polyphase components satisfy (66) and (67).

Lemma 2 The filters $\{h_0, h_1, h_2\}$ with symmetries (57-59) satisfy the perfect reconstruction conditions if their polyphase components are given by

$$H_{00}(z) = \frac{z^{-1}}{\sqrt{2}}\, A^2(z) + \frac{1}{\sqrt{2}}\, B^2(z) \tag{68}$$

$$H_{10}(z) = \frac{z^{-1}}{\sqrt{2}}\, A^2(z) - \frac{1}{\sqrt{2}}\, B^2(z) \tag{69}$$

$$H_{20}(z) = \sqrt{2}\, A(z)\, B(z) \tag{70}$$

with (60-62) where $A(z)$ and $B(z)$ satisfy

$$A(z)\, A(1/z) + B(z)\, B(1/z) = 1. \tag{71}$$

4.1 Example 2

In the following example, $H_0(z)$ will have $(1+z^{-1})^5$ as a factor and each wavelet will have three vanishing moments. This example is identical to Example 5.6 in [11], however, here the lowpass filter $H_0(z)$ is obtained first by asking that it satisfy (65). As in Example 1, we will use a linear combination of two maximally-flat lowpass filters $F^{(m,l)}(z)$ to obtain a minimal-length filter $H_0(z)$ satisfying (65). Specifically, we set

$$H_0(z) := z^{-5}\,\sqrt{2}\,\left(\alpha\,F^{(2,2)}(z) + (1-\alpha)\,F^{(3,2)}(z)\right). \tag{72}$$

Then, for special values of α, $H_0(z)$ satisfies (65). From (46) we have

$$F^{(2,2)}(z) = \frac{z^2}{2^{12}}\,(1+z^{-1})^5\,(35\,z^2 - 220\,z + 498 - 220\,z^{-2} + 35\,z^{-2})$$

$$F^{(3,2)}(z) = \frac{z^3}{2^{14}}\,(1+z^{-1})^7\,(63\,z^2 - 364\,z + 730 - 364\,z^{-2} + 63\,z^{-2}).$$

For convenience, we can write $H_{00}(z)$ as

$$H_{00}(z) = \alpha\,G_1(z) + (1-\alpha)\,G_0(z)$$

where $G_0(z)$ and $G_1(z)$ are appropriately defined. Following the procedure of Example 1, the values of α are sought for which all the zeros of $1 - 2\,H_{00}(z)\,H_{00}(1/z)$ are of even multiplicity. Using the change of variables used in Example 1, we obtain the polynomial,

$$\frac{21}{2^{16}}\,x^3\,\left[(1408 + 396\,x + 231\,x^2) - \alpha\,(768 + 96\,x + 42\,x^2) - \alpha^2\,189\,x^2\right]$$

or

$$x^3\,\left(P_0(\alpha) + P_1(\alpha)\,x + P_2(\alpha)\,x^2\right). \tag{73}$$

The term x^3 in (73) implies that $1 - 2\,H_{00}(z)\,H_{00}(1/z)$ has a root at $z = 1$ of multiplicity 6. The remaining roots of (73) must be of even degree. Therefore, we can find α by setting the discriminant, $D(\alpha) = P_1^2(\alpha) - 4\,P_0(\alpha)\,P_2(\alpha)$, to zero. The discriminant is a third degree polynomial in α:

$$D(\alpha) = \frac{3^3 \cdot 7^2}{2^{28}}\,(12\,\alpha - 11)\,(1008\,\alpha^2 - 716\,\alpha - 2167).$$

The roots of $D(\alpha)$ are approximately $\{-1.15346,\ 1.86378,\ 0.91667\}$. Setting α to the most negative of the three roots produces the smoothest scaling function, which is also given in Figure 3 of [11].

For case II, we do not know how to obtain the polynomials $A(z)$ and $B(z)$ directly from the lowpass filter $H_0(z)$ using a simple root selection procedure as in case I. However, $H_1(z)$ and $H_2(z)$ can be obtained using the methods of [2, 13, 18]. The wavelet filters $h_1(n)$ and $h_2(n)$, are the same as in Example 5.6 of [11]. The corresponding anti-symmetric wavelets, illustrated in Figure 3 of [11], have three vanishing moments each. They are supported on $[0, 11]$ and $[0, 10]$ and satisfy $\psi_1(t) = -\psi_1(11 - t)$ and $\psi_2(t) = -\psi_2(10 - t)$. Note that the lengths of the two wavelets are different, as expected from [11].

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