MULTIWAVELET BASES WITH EXTRA APPROXIMATION PROPERTIES

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ABSTRACT

Because multiwavelet bases normally lack important properties traditional wavelet bases (of equal approximation order) possess, the discrete multiwavelet transform is less useful for signal processing, unless preceded by a preprocessing step. This paper examines properties of orthogonal multiwavelet bases, with approximation order > 1, that possess those properties normally absent. For these "balanced" bases (after Lebrun and Vetterli) prefiltering can be avoided. Using results regarding $M-$ band wavelet bases, it has been found that balanced multiwavelet bases can be characterized in terms of the divisibility of certain transfer functions by powers of $(z^{-2r} - 1)/(z^{-1} - 1)$. The paper also presents a balanced version of the DGHM basis — the scaling functions are simultaneously symmetric, orthogonal and of compact support.

1. INTRODUCTION

Multiwavelet bases (wavelet bases based on multiple scaling functions) have been investigated for several years now. With this generalization, it is possible to construct orthogonal (real-valued) bases for which the scaling functions have compact support, approximation order greater than 1, and symmetry, which is not possible with traditional wavelet bases. This is demonstrated by the basis of Donovan, Geromino, Hardin and Massopust [2, 3].

It must be realized however, that traditional wavelet bases have several attributes that multiwavelet bases, of equal approximation order, do not normally possess. Consequently, a prefiltering step is fundamentally more important for the discrete multiwavelet transform (DMWT), than is so for the traditional DWT. Accordingly, several authors have addressed the design of prefilters specifically for the DMWT [5, 13, 14]. One the other hand, it is also natural to investigate the design of multiwavelets, with specified approximation order, that *do* possess those attributes normally absent. Lebrun and Vetterli addressed this topic and in [7, 8] coined the term "balanced" multiwavelets.

The paper will examine balanced orthogonal multiwavelets and characterize them in terms of the scaling filter coefficients. For an orthogonal multiwavelet basis to be balanced (up to its approximation order), certain transfer functions must be divisible by powers of $(z^{-2r} - 1)/(z^{-1} - 1)$. The same divisibility condition arises in the case of $M-$band wavelet bases [6, 12].

This paper also presents a version of the DGHM basis that is balanced up to it approximation order $K = 2$; the scaling functions are simultaneously symmetric, orthogonal and of compact support.

2. PRELIMINARIES

A multiwavelet basis (of multiplicity r) is characterized by r scaling functions and r wavelet functions. The scaling space $\mathcal{V}_j$ is defined by

$$\mathcal{V}_j = \operatorname*{Span}_k \{\phi_0(2^j t - k), \ldots, \phi_{r-1}(2^j t - k)\}$$

and the wavelet space $\mathcal{W}_j$ is defined by

$$\mathcal{W}_j = \operatorname*{Span}_k \{\psi_0(2^j t - k), \ldots, \psi_{r-1}(2^j t - k)\}.$$

For wavelet bases based on a single scaling function (traditional wavelet bases), from the nesting condition $\mathcal{V}_j \subset \mathcal{V}_{j+1}$ the dilation equation is obtained. The same is true for multiwavelets:

$$\underline{\phi}(t) = \sqrt{2} \sum_n C_0(n)\, \underline{\phi}(2t - n)$$

is the matrix dilation equation where $\underline{\phi}(t) = (\phi_0(t), \phi_1(t))^t$ for a multiplicity $r = 2$ multiwavelet basis. The sequence $C_0(n)$ is the matrix scaling filter, an $r-$input $r-$output filter. The notation for $C_0(n)$ used in this paper is

$$[C_0(n)]_{i,j} = h_i(nr + j).$$

For example, when $r = 2$,

$$C_0(0) = \begin{pmatrix} h_0(0) & h_0(1) \\ h_1(0) & h_1(1) \end{pmatrix}, \quad C_0(1) = \begin{pmatrix} h_0(2) & h_0(3) \\ h_1(2) & h_1(3) \end{pmatrix},$$

etc. In this case, there are two scaling filters $h_0(n)$ and $h_1(n)$. In general there are r scaling filters, and r wavelet filters. The transfer functions $H_i(z)$ are given by

$$H_i(z) = \sum_n h_i(n) z^{-n}.$$

2.1. Multiwavelet Filter Bank

As noted by Rieder *et al* [9, 10], the filter bank associated with a multiwavelet basis can be drawn as a 4-channel *scalar*-valued filter bank in Figure 1. In general, the 2-channel $r-$vector filter bank associated with a multiwavelet basis of multiplicity r can be reorganized as a $2r-$ channel scalar filter bank.

It follows that the orthogonality conditions for the multiwavelet filter bank are the same as the orthogonality conditions for a $2r-$channel scalar filter bank. Here lies the similarity between multiplicity r multiwavelet filter banks and $2r-$channel filter banks, the later being associated with $M-$band wavelet bases [6, 12]. The difference between these two systems lies in the way the filter bank is iterated on the lowpass branch. For 4-band wavelet systems (1 scaling function with 3 wavelet functions), the filter bank is applied again only to the first (lowpass) channel. In the multiwavelet case, the outputs of the first two channels are interlaced to form a single

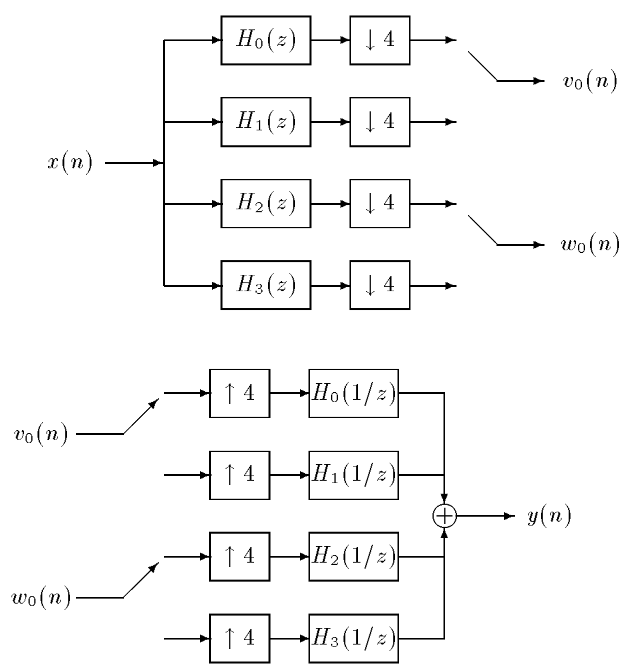

Figure 1: Filter bank associated with a ($r = 2$) multiwavelet basis, drawn as a 4-band scalar filter bank with interlacing/de-interlacing.

scalar sequence, and the filter bank is repeated on that single scalar sequence.

Unfortunately, the filter bank associated with a multiwavelet basis does not inherit the zero moment properties of the basis, unless the basis satisfies additional properties. In other words, for multiwavelet bases, zero moments of the wavelets $\psi_i(t)$ defined on $\mathbb{R}$, do not imply zero moments of the wavelet filters $h_i(n)$ defined on $\mathbb{Z}$. Multiwavelet bases for which the zero moment properties *do* carry over to the discrete-time filter bank are called *balanced* after Lebrun and Vetterli [7, 8].

2.2. Traditional 2-Channel Case

Consider first the iterated 2-channel filter bank associated with the traditional orthogonal 2-band wavelet basis, illustrated in Figure 2. Suppose the basis has approximation order K: $\int t^k \psi(t) dt = 0$, for $k = 0, \ldots, K-1$. It follows, through the dilation equation, that if the input $x(n)$ is a polynomial of degree $k < K$, $x(n) \in \mathcal{P}_{K-1}$, then $w_0(n) = 0$ and $v_0(n) \in \mathcal{P}_{K-1}$. The lowpass and highpass channels respectively preserve/annihilate polynomials. Consequently, $w_i(n) = 0$ for all $i \geq 0$. Except for the lowpass channel, all channels of the iterated filter bank annihilate polynomials of degree $k < K$. This depends on *both* the preservation of $\mathcal{P}_{K-1}$ by the lowpass branch *and* the annihilation of $\mathcal{P}_{K-1}$ by the highpass branch.

2.3. The Multiwavelet Case

Consider now the multiwavelet case. Associated with a multiplicity $r = 2$ multiwavelet basis are 2 scaling filters $h_0(n), h_1(n)$ and 2 wavelet filters $h_2(n), h_3(n)$. The multiwavelet filter bank is shown in Figure 1. Note that the outputs of the 2 scaling filters are interlaced.

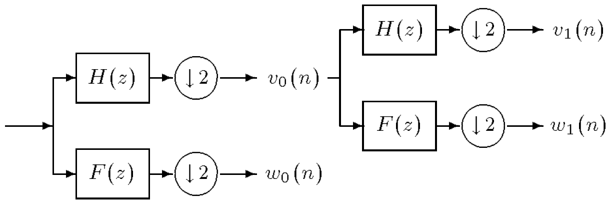

Figure 2: Iterated filter bank.

Suppose the multiwavelet basis has approximation order K:

$$\int t^k \psi_i(t)\, dt = 0 \qquad k = 0, \ldots, K-1; \qquad i = 0, 1.$$

If the input $x(n)$ is a polynomial of degree $k < K$, then generally $v_0(n)$ is not in $\mathcal{P}_{K-1}$ and $w_0(n) \neq 0$. The multiwavelet filter bank neither preserves nor annihilates $\mathcal{P}_{K-1}$. In the multiwavelet case, for the preservation/annihilation of polynomials, it is not sufficient that the moments of $\psi_i(t)$ vanish.

Multiwavelet bases for which the associated filter bank *does* preserve/annihilate polynomials of degree $k < K$ are said to be *order-K balanced*.

2.4. Prefiltering

For wavelet transforms based on a single scaling function, the preprocessing of the discrete data, ("prefiltering" or "wavelet initialization") is often omitted without consequence. However, because the multiwavelet filter bank does not inherit the moment properties of the basis, a prefiltering step is essential when using a multiwavelet transform. Although prefiltering alleviates some of the problems, some reasons for wishing to avoid prefiltering remain:

1. If the prefiltering step does not constitute an orthogonal transform, then the orthogonality of the DMWT is "lost".
2. If the prefilter is not symmetric, then the symmetry properties of the DMWT are "lost".
3. The use of prefiltering effectively increases the support of the basis functions, if the prefilter is more than a single coefficient.

In light of these issues, it is natural to pursue the design of balanced multiwavelet bases. To obtain balanced bases, a characterization of them in terms of the scaling filters will be convenient.

3. BALANCING CONDITIONS

From [11], the condition for order-1 balancing, for multiplicity $r = 2$ multiwavelet bases is

$$(z^{-3} + z^{-2} + z^{-1} + 1) \quad \text{divides} \quad H_0(z) + H_1(z).$$

Order-1 balanced multiwavelet filter banks preserve/annihilate constant signals.

From [11], the condition for order-2 balancing, for multiplicity $r = 2$ multiwavelet bases is

$$(z^{-3}+z^{-2}+z^{-1}+1)^2 \quad \text{divides} \quad H_0(z) + \left(\frac{3 - z^{-4}}{2}\right) H_1(z).$$

Order-2 balanced multiwavelets filter banks preserve/annihilate ramp and constant signals.

From [11], the condition for order-K balancing for multiplicity-r multiwavelet bases for general K and r, is

$$\left(\frac{z^{-2r}-1}{z^{-1}-1}\right)^{K} \quad \text{divides} \quad \sum_{i=0}^{r-1} Q_i(z^{2r})\, H_i(z) \qquad (1)$$

where

$$Q_i(z) = \frac{P}{r^{(K-1)}\,(K-1)!} \sum_{l=0}^{K-1} \frac{(-1)^l}{(i+l\,r)} \binom{K-1}{l} z^{-l} \qquad (2)$$

and

$$P = i\cdot(i+r)\cdot(i+2r)\cdots(i+(K-1)\cdot r). \qquad (3)$$

In Table 1 we enumerate the condition for the first few K for the special case of $r = 2$ scaling functions. Note that in the special case $r = 1$, one retrieves the usual moment condition: $(z^{-1}+1)^K$ divides $H_0(z)$.

4. EXAMPLES

4.1. Short Support

In this example we consider orthogonal multiwavelet bases of approximation order 2, balanced up to their approximation order. Certainly Daubechies' basis D_2 provides such a solution. In that case $h_1(n) = h_0(n-2)$, $\phi_0(t) = \phi_{D2}(2t)$, and $\phi_1(t) = \phi_{D2}(2t-1)$. For that solution, the length of $h_0(n)$ and $h_1(n)$ are both 4. However, there exists a balanced solution for which the length of $h_0(n)$ is less than 4, when the length of $h_1(n)$ is allowed to exceed 4. With $h_0(n)$ supported on $\{1,2,3\}$, the shortest length $h_1(n)$, for which a solution was obtained, was of length 8. The solution was obtained by forming the quadratic orthogonality constraints on $h_i(n)$ and the additional balancing conditions. The solutions to this system of nonlinear equations were then obtained using a lexical Gröbner basis [1] (for the computation of which, the software *Singular* was employed [4]). The coefficients can be found in [11]. The filter bank associated with this order-2 balanced multiwavelet basis preserves/annihilates ramp and constant signals.

4.2. Balanced DGHM-Like Multiwavelet Bases

It is often desired, in image processing for example, that the scaling function of a wavelet basis be symmetric and have compact support. However, as is well known, except for the Haar basis, the scaling function $\phi(t)$ of an orthogonal wavelet basis can not have both symmetry and compact support. For this reason, the multiwavelet basis constructed by Donovan, Geromino, Hardin and Massopust [2, 3] is most interesting. Their orthogonal basis is based on 2 scaling functions and 2 wavelet functions, all of which have symmetry and compact support. The DGHM basis has order-2 approximation (2 moments of the wavelets vanish). However, it has order-0 balancing; the filter bank associated with the DGHM basis does not even preserve/annihilate constant signals.

This section describes a DGHM-like basis balanced up to its approximation order $K = 2$. Like the DGHM basis

1. Both $\phi_0(t)$ and $\phi_1(t)$ are symmetric.
2. Both $h_0(n)$ and $h_1(n)$ are symmetric odd length (Type 1) FIR filters.
3. $h_0(n)$ and $h_1(n)$ differ in length by 4.

The DGHM scaling filters $h_0(n)$, $h_1(n)$ in [2, 3] are of lengths 3 and 7. Because balancing is a stronger condition than is approximation order, it can not be expected that order-2 balanced GHM-like scaling filters are possible for these lengths. For these lengths (3,7), it is possible to increase the balance order by one, and decrease the approximation order by 1. The result is a system with order-1 approximation and order-1 balancing, shown in Figure 4.

By increasing the lengths to (7,11), it is possible to obtain a DGHM-like multiwavelet basis having oder-2 approximation and order-2 balancing, as illustrated in Figure 5. For the scaling filter coefficients, an algebraic solution in terms of radicals was obtained, again using Gröbner bases. Note that the 2 scaling functions are very similar to one another. It is also interesting to note that the order-2 balanced DGHM-like basis possesses cusps. In the traditional 2-channel case, increasing the approximation order generally increases the smoothness of the basis functions. Apparently, for balanced DGHM-like bases, a different behavior emerges.

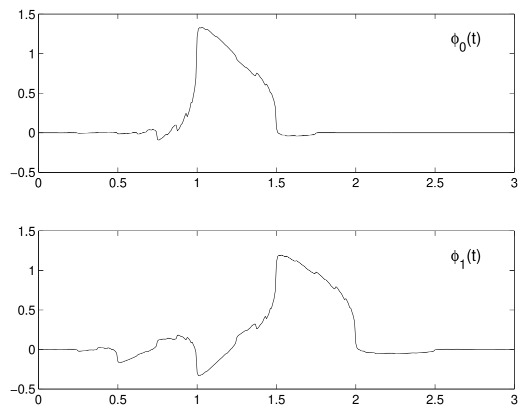

Figure 3: Order-2 balanced multiwavelet scaling functions. The supports of $\phi_0(t)$ and $\phi_1(t)$ are $[0,2]$ and $[0,3]$.

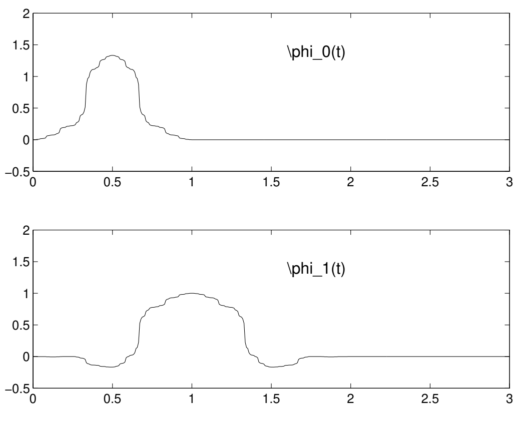

Figure 4: Symmetric order-1 balanced multiwavelet scaling functions. $\phi_0(t)$ is symmetric around $t = 0.5$. $\phi_1(t)$ is symmetric around $t = 1$. Accordingly, their supports are $[0,1]$ and $[0,2]$.

Table 1: Conditions on scaling filters $h_0(n)$ and $h_1(n)$ for order-K balancing, for $r = 2$.

For order-1 balancing:		
$(z^{-3} + z^{-2} + z^{-1} + 1)$	divides	$H_0(z) + H_1(z)$.
For order-2 balancing:		
$(z^{-3} + z^{-2} + z^{-1} + 1)^2$	divides	$H_0(z) + \left(\frac{3 - z^{-4}}{2}\right) H_1(z)$.
For order-3 balancing:		
$(z^{-3} + z^{-2} + z^{-1} + 1)^3$	divides	$H_0(z) + \left(\frac{15 - 10\, z^{-4} + 3\, z^{-8}}{8}\right) H_1(z)$.
For order-4 balancing:		
$(z^{-3} + z^{-2} + z^{-1} + 1)^4$	divides	$H_0(z) + \left(\frac{35 - 35\, z^{-4} + 21\, z^{-8} - 5\, z^{-12}}{16}\right) H_1(z)$.

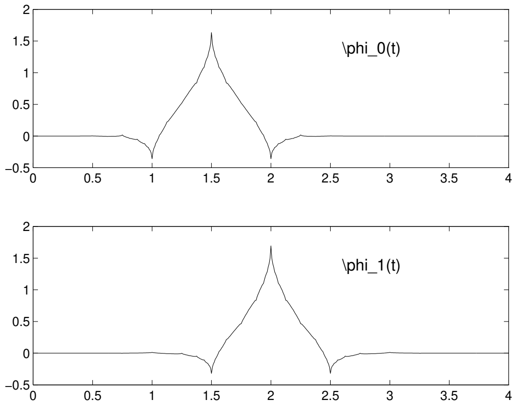

Figure 5: Symmetric order-2 balanced multiwavelet scaling functions. $\phi_0(t)$ is symmetric around $t = 1.5$. $\phi_1(t)$ is symmetric around $t = 2$. Accordingly, their supports are $[0, 3]$ and $[0, 4]$.

5. CONCLUSION

This paper highlighted important differences between traditional wavelet and multiwavelet bases with equal approximation order. It was noted that multiwavelet bases lack properties traditional wavelet bases possess. Those properties are important for processing discrete-time signals; with out them, the associated filter bank fails to give efficient representations of smooth signals. Although this problem can be partly overcome with appropriate prefilters, the paper described multiwavelets that do possess the desired properties. In particular, a characterization of order-K balanced orthogonal multiplicity-r multiwavelets was given in terms of the divisibility of a certain transfer function by powers of $(z^{-2r} - 1)/(z^{-1} - 1)$. The paper also presented a balanced version of the DGHM basis — the scaling functions are simultaneously symmetric, orthogonal and of compact support.

6. REFERENCES

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