LINEAR-PHASE FIR FILTERS

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If the real and imaginary parts of $H^f(\omega)$ are given by

$$H^f(\omega) = R(\omega) + j \cdot I(\omega)$$

the magnitude and phase are defined as

$$|H^f(\omega)| = \sqrt{R^2(\omega) + I^2(\omega)}$$

$$p(\omega) = \arctan\left(\frac{I(\omega)}{R(\omega)}\right)$$

so that

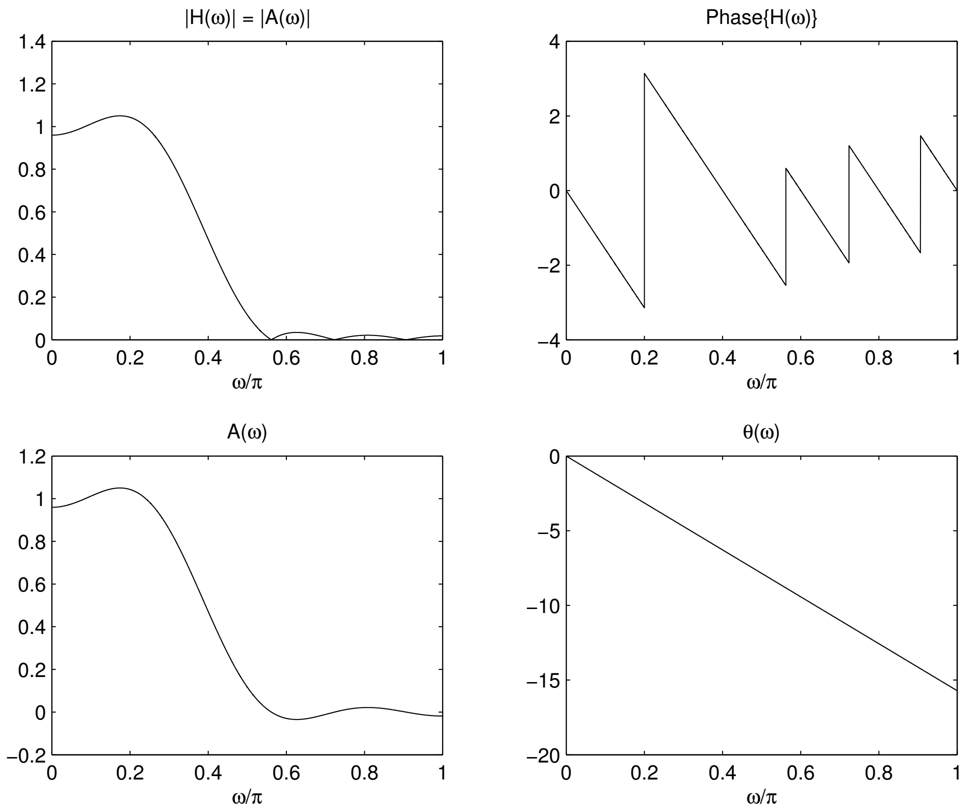
$$H^f(\omega) = |H^f(\omega)| \cdot e^{jp(\omega)}.$$

With this definition, $|H^f(\omega)|$ is never negative and $p(\omega)$ is usually discontinuous, but it can be very helpful to write $H^f(\omega)$ as

$$H^f(\omega) = A(\omega) \cdot e^{j\theta(\omega)}$$

where $A(\omega)$ can be positive and negative, and $\theta(\omega)$ continuous. $A(\omega)$ is called the *amplitude* response. The figure illustrates the difference between $|H^f(\omega)|$ and $A(\omega)$.

THE AMPLITUDE RESPONSE (2)



A linear-phase phase filter is one for which the continuous phase $\theta(\omega)$ is linear.

$$H^f(\omega) = A(\omega)\, e^{j\theta(\omega)}$$

with

$$\theta(\omega) = -M \cdot \omega + B.$$

We assume in the following that the impulse response $h(n)$ is real-valued.

WHY LINEAR-PHASE?

If a discrete-time cosine signal

$$x_1(n) = \cos(\omega_1 n + \phi_1)$$

is processed through a discrete-time filter with frequency response

$$H^f(\omega) = A(\omega) \cdot e^{j\theta(\omega)}$$

then the output signal is given by

$$y_1(n) = A(\omega_1) \cos(\omega_1 n + \phi_1 + \theta(\omega_1))$$

or

$$y_1(n) = A(\omega_1) \cos\left(\omega_1 \left(n + \frac{\theta(\omega_1)}{\omega_1}\right) + \phi_1\right).$$

The LTI system has the effect of scaling the cosine signal and delaying it by $-\theta(\omega_1)/\omega_1$.

When does the system delay cosine signals with different frequencies by the same amount?

$$\Longrightarrow \qquad \frac{\theta(\omega)}{\omega} = \text{constant}$$

$$\Longrightarrow \qquad \theta(\omega) = K\,\omega$$

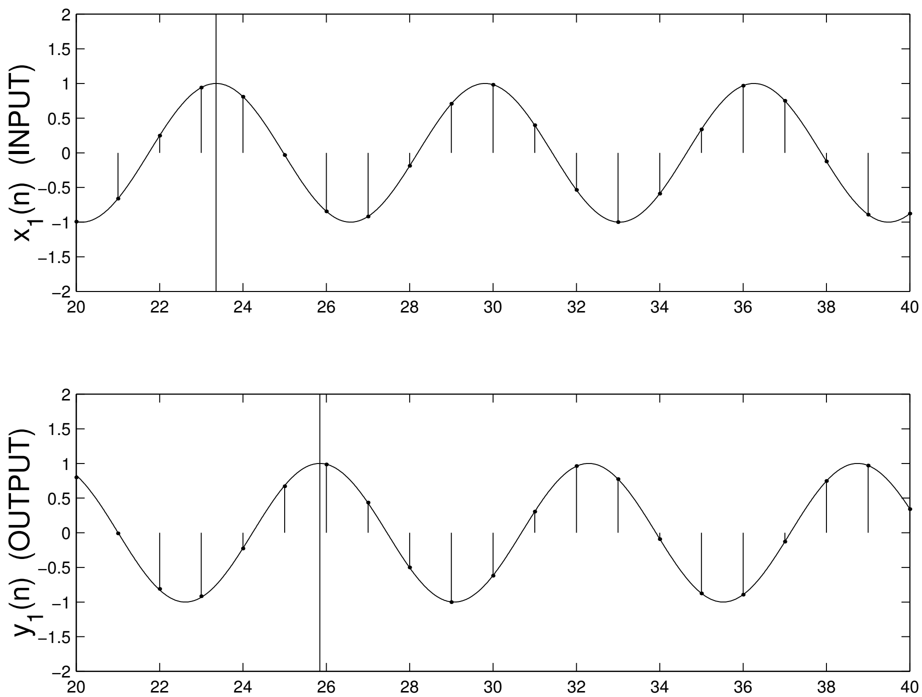
$\Longrightarrow$ The phase is linear

The function $\theta(\omega)/\omega$ is called the *phase delay*. A linear phase filter therefore has constant phase delay.

Consider an discrete-time filter described by the difference equation

$$y(n) = -\,0.1821\,x(n) + 0.7865\,x(n-1) - 0.6804\,x(n-2) + x(n-3) + 0.6804\,y(n-1) - 0.7865\,y(n-2) + 0.1821\,y(n-3).$$

When $\omega_1 = 0.31\,\pi$, then the delay is $-\theta(\omega_1)/\omega_1 = 2.45$.
The delay is illustrated in the figure:



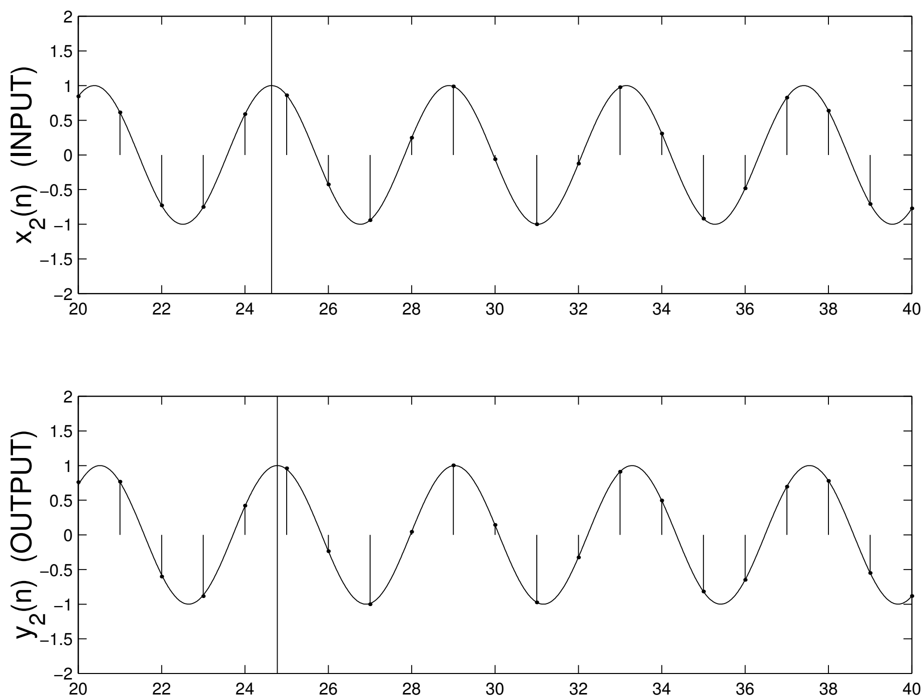
Notice that the delay is fractional — the discrete-time samples are not exactly reproduced in the output.

The fractional delay can be interpreted in this case as a delay of the underlying continuous-time cosine signal.

WHY LINEAR-PHASE: EXAMPLE (2)

Consider the same system given on the previous slide, but let us change the frequency of the cosine signal.

When $\omega_2 = 0.47\,\pi$, then the delay is $-\theta(\omega_2)/\omega_2 = 0.14$.



For this example, the delay depends on the frequency, because this system does not have linear phase.

WHY LINEAR-PHASE: MORE

From the previous slides, we see that a filter will delay different frequency components of a signal by the same amount if the filter has linear phase (constant phase delay).

In addition, when a narrow band signal (as in AM modulation) goes through a filter, the envelop will be delayed by the *group delay* or *envelop delay* of the filter. The amount by which the envelop is delayed is independent of the carrier frequency only if the filter has linear phase. (See page 214 in Mitra.)

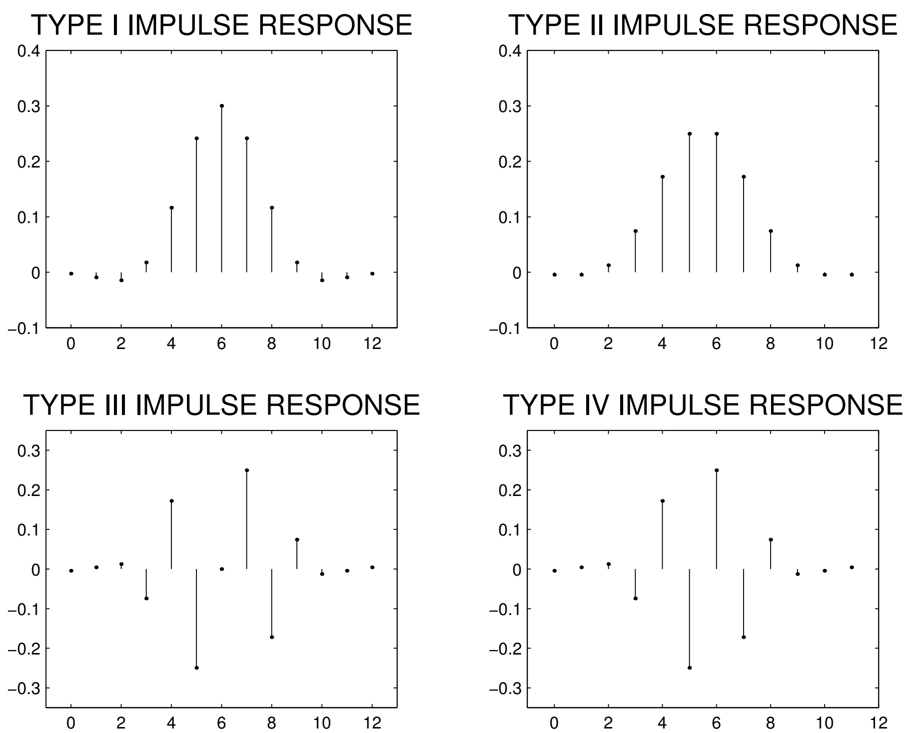
Also, in applications like image processing, filters with non-linear phase can introduce artifacts that are visually annoying.

FOUR TYPES OF LINEAR-PHASE FIR FILTERS

Sec 4.4.3 in Mitra

Linear-phase FIR filter can be divided into four basic types.

Type	impulse response	
I	symmetric	length is odd
II	symmetric	length is even
III	anti-symmetric	length is odd
IV	anti-symmetric	length is even



When $h(n)$ is nonzero for $0 \le n \le N-1$ (the length of the impulse response $h(n)$ is N), then the symmetry of the impulse response can be written as

$$h(n) = h(N-1-n)$$

and the anti-symmetry can be written as

$$h(n) = -h(N-1-n).$$

FOUR TYPES OF LINEAR-PHASE FIR FILTERS

Important note: If the impulse response $h(n)$ is *complex*-valued, then to have linear-phase the impulse response should be *conjugate*-symmetric or *conjugate*-anti-symmetry.

TYPE I: ODD-LENGTH SYMMETRIC

The frequency response of a length $N = 5$ FIR Type I filter can be written as follows.

$$H^f(\omega) = h_0 + h_1 e^{-j\omega} + h_2 e^{-2j\omega} + h_1 e^{-3j\omega} + h_0 e^{-4\omega} \tag{1}$$

$$= e^{-2j\omega}\left(h_0 e^{2j\omega} + h_1 e^{j\omega} + h_2 + h_1 e^{-j\omega} + h_0 e^{-2j\omega}\right) \tag{2}$$

$$= e^{-2j\omega}\left(h_0(e^{2j\omega} + e^{-2j\omega}) + h_1(e^{j\omega} + e^{-j\omega}) + h_2\right) \tag{3}$$

$$= e^{-2j\omega}\left(2h_0 \cos(2\omega) + 2h_1 \cos(\omega) + h_2\right) \tag{4}$$

$$= A(\omega) e^{j\theta(\omega)} \tag{5}$$

where

$$\theta(\omega) = -2\omega, \qquad A(\omega) = 2h_0 \cos(2\omega) + 2h_1 \cos(\omega) + h_2.$$

Note that $A(\omega)$ is real-valued and can be both positive and negative. In general, for a Type I FIR filters of length N:

$$H^f(\omega) = A(\omega) e^{j\theta(\omega)}$$

$$\boxed{A(\omega) = h(M) + 2\sum_{n=0}^{M-1} h(n) \cos((M-n)\omega).}$$

$$\boxed{\theta(\omega) = -M\omega}$$

$$\boxed{M = \frac{N-1}{2}.}$$

The frequency response of a length $N = 4$ FIR Type II filter can be written as follows.

$$\begin{aligned} H^f(\omega) &= h_0 + h_1 e^{-j\omega} + h_1 e^{-2j\omega} + h_0 e^{-3j\omega} && (6) \\ &= e^{-\frac{3}{2}j\omega}\left(h_0 e^{\frac{3}{2}j\omega} + h_1 e^{\frac{1}{2}j\omega} + h_1 e^{-\frac{1}{2}j\omega} + h_0 e^{-\frac{3}{2}j\omega}\right) && (7) \\ &= e^{-\frac{3}{2}j\omega}\left(h_0(e^{\frac{3}{2}j\omega} + e^{-\frac{3}{2}j\omega}) + h_1(e^{\frac{1}{2}j\omega} + e^{-\frac{1}{2}j\omega})\right) && (8) \\ &= e^{-\frac{3}{2}j\omega}\left(2h_0\cos\left(\tfrac{3}{2}\omega\right) + 2h_1\cos\left(\tfrac{1}{2}\omega\right)\right) && (9) \\ &= A(\omega)e^{j\theta(\omega)} && (10) \end{aligned}$$

where

$$\theta(\omega) = -\frac{3}{2}\omega, \qquad A(\omega) = 2h_0\cos\left(\tfrac{3}{2}\omega\right) + 2h_1\cos\left(\tfrac{1}{2}\omega\right).$$

In general, for a Type II FIR filters of length N:

$$H^f(\omega) = A(\omega)e^{j\theta(\omega)}$$

$$\boxed{A(\omega) = 2\sum_{n=0}^{\frac{N}{2}-1} h(n)\cos\left((M-n)\omega\right)}$$

$$\boxed{\theta(\omega) = -M\omega}$$

$$\boxed{M = \frac{N-1}{2}.}$$

TYPE III: ODD-LENGTH ANTI-SYMMETRIC

The frequency response of a length $N = 5$ FIR Type III filter can be written as follows.

$$
\begin{aligned}
H^f(\omega) &= h_0 + h_1 e^{-j\omega} - h_1 e^{-3j\omega} - h_0 e^{-4\omega} && (11)\\
&= e^{-2j\omega}\left(h_0 e^{2j\omega} + h_1 e^{j\omega} - h_1 e^{-j\omega} - h_0 e^{-2j\omega}\right) && (12)\\
&= e^{-2j\omega}\left(h_0(e^{2j\omega} - e^{-2j\omega}) + h_1(e^{j\omega} - e^{-j\omega})\right) && (13)\\
&= e^{-2j\omega}\left(2jh_0 \sin(2\omega) + 2jh_1 \sin(\omega)\right) && (14)\\
&= e^{-2j\omega} j \left(2h_0 \sin(2\omega) + 2h_1 \sin(\omega)\right) && (15)\\
&= e^{-2j\omega} e^{j\frac{\pi}{2}} \left(2h_0 \sin(2\omega) + 2h_1 \sin(\omega)\right) && (16)\\
&= A(\omega) e^{j\theta(\omega)} && (17)
\end{aligned}
$$

where

$$\theta(\omega) = -2\omega + \frac{\pi}{2}, \qquad A(\omega) = 2h_0 \sin(2\omega) + 2h_1 \sin(\omega).$$

In general, for a Type III FIR filters of length N:

$$H^f(\omega) = A(\omega) e^{j\theta(\omega)}$$

$$\boxed{A(\omega) = 2\sum_{n=0}^{M-1} h(n) \sin((M-n)\omega)}$$

$$\boxed{\theta(\omega) = -M\omega + \frac{\pi}{2}}$$

$$\boxed{M = \frac{N-1}{2}.}$$

TYPE IV: EVEN-LENGTH ANTI-SYMMETRIC

The frequency response of a length $N = 4$ FIR Type IV filter can be written as follows.

$$
\begin{aligned}
H^f(\omega) &= h_0 + h_1 e^{-j\omega} - h_1 e^{-2j\omega} - h_0 e^{-3j\omega} && (18)\\
&= e^{-\frac{3}{2}j\omega}\left(h_0 e^{\frac{3}{2}j\omega} + h_1 e^{\frac{1}{2}j\omega} - h_1 e^{-\frac{1}{2}j\omega} - h_0 e^{-\frac{3}{2}j\omega}\right) && (19)\\
&= e^{-\frac{3}{2}j\omega}\left(h_0(e^{\frac{3}{2}j\omega} - e^{-\frac{3}{2}j\omega}) + h_1(e^{\frac{1}{2}j\omega} - e^{-\frac{1}{2}j\omega})\right) && (20)\\
&= e^{-\frac{3}{2}j\omega}\left(2jh_0 \sin\left(\tfrac{3}{2}\omega\right) + 2jh_1 \sin\left(\tfrac{1}{2}\omega\right)\right) && (21)\\
&= e^{-\frac{3}{2}j\omega} j\left(2h_0 \sin\left(\tfrac{3}{2}\omega\right) + 2h_1 \sin\left(\tfrac{1}{2}\omega\right)\right) && (22)\\
&= e^{-\frac{3}{2}j\omega} e^{j\frac{\pi}{2}}\left(2h_0 \sin\left(\tfrac{3}{2}\omega\right) + 2h_1 \sin\left(\tfrac{1}{2}\omega\right)\right) && (23)\\
&= A(\omega) e^{j\theta(\omega)} && (24)
\end{aligned}
$$

where

$$
\theta(\omega) = -\frac{3}{2}\omega + \frac{\pi}{2}, \qquad A(\omega) = 2h_0 \sin\left(\tfrac{3}{2}\omega\right) + 2h_1 \sin\left(\tfrac{1}{2}\omega\right).
$$

In general, for a Type IV FIR filters of length N:

$$
H^f(\omega) = A(\omega)\, e^{j\theta(\omega)}
$$

$$
\boxed{A(\omega) = 2 \sum_{n=0}^{\frac{N}{2}-1} h(n) \sin\left((M-n)\omega\right)}
$$

$$
\boxed{\theta(\omega) = -M\omega + \frac{\pi}{2}}
$$

$$
\boxed{M = \frac{N-1}{2}.}
$$

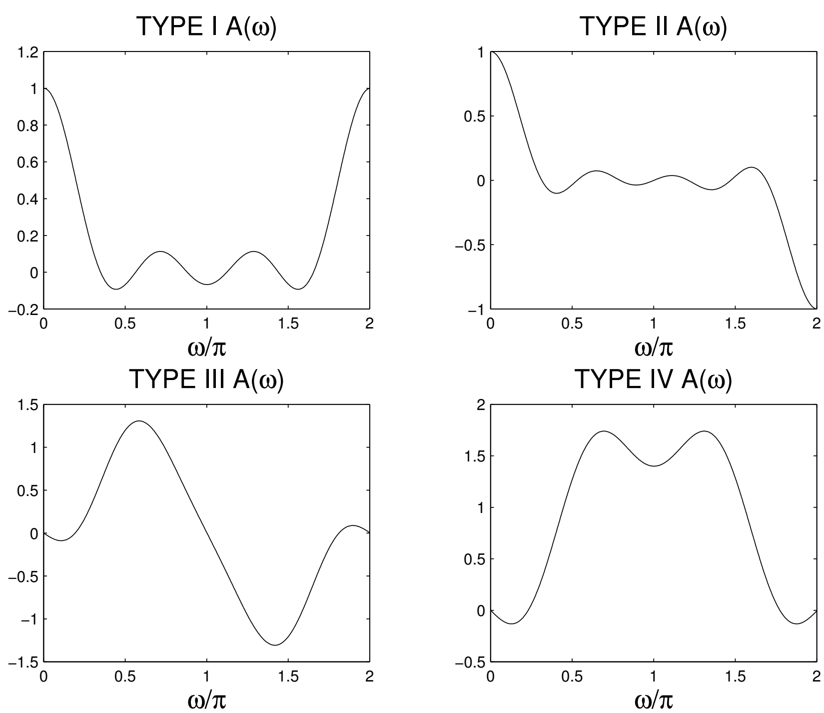
Type	$\theta(\omega)$	$A(\omega)$
I	$-M\omega$	$h(M) + 2\sum_{n=0}^{M-1} h(n)\cos((M-n)\omega)$
II	$-M\omega$	$2\sum_{n=0}^{\frac{N}{2}-1} h(n)\cos((M-n)\omega)$
III	$-M\omega + \frac{\pi}{2}$	$2\sum_{n=0}^{M-1} h(n)\sin((M-n)\omega)$
IV	$-M\omega + \frac{\pi}{2}$	$2\sum_{n=0}^{\frac{N}{2}-1} h(n)\sin((M-n)\omega)$

$$M = \frac{N-1}{2}$$

AMPLITUDE RESPONSE CHARACTERISTICS

To analyze or design linear-phase FIR filters, we need to know the characteristics of the amplitude response $A(\omega)$.

Type	Properties	
I	$A(\omega)$ is even about $\omega = 0$	$A(\omega) = A(-\omega)$
	$A(\omega)$ is even about $\omega = \pi$	$A(\pi + \omega) = A(\pi - \omega)$
	$A(\omega)$ is periodic with 2π	$A(\omega + 2\pi) = A(\omega)$
II	$A(\omega)$ is even about $\omega = 0$	$A(\omega) = A(-\omega)$
	$A(\omega)$ is odd about $\omega = \pi$	$A(\pi + \omega) = -A(\pi - \omega)$
	$A(\omega)$ is periodic with 4π	$A(\omega + 4\pi) = A(\omega)$
III	$A(\omega)$ is odd about $\omega = 0$	$A(\omega) = -A(-\omega)$
	$A(\omega)$ is odd about $\omega = \pi$	$A(\pi + \omega) = -A(\pi - \omega)$
	$A(\omega)$ is periodic with 2π	$A(\omega + 2\pi) = A(\omega)$
IV	$A(\omega)$ is odd about $\omega = 0$	$A(\omega) = -A(-\omega)$
	$A(\omega)$ is even about $\omega = \pi$	$A(\pi + \omega) = A(\pi - \omega)$
	$A(\omega)$ is periodic with 4π	$A(\omega + 4\pi) = A(\omega)$



The frequency response $H^f(\omega)$ of an FIR filter can be evaluated at L equally spaced frequencies between 0 and π using the DFT. Consider a causal FIR filter with an impulse response $h(n)$ of length-N, with $N \leq L$. Samples of the frequency response of the filter can be written as

$$H\left(\frac{2\pi}{L}k\right) = \sum_{n=0}^{N-1} h(n) e^{-j\frac{2\pi}{L}nk}.$$

Define the L-point signal $\{g(n), 0 \leq n \leq L-1\}$ as

$$g(n) = \begin{cases} h(n) & 0 \leq n \leq N-1 \\ 0 & N \leq n \leq L-1. \end{cases}$$

Then

$$H\left(\frac{2\pi}{L}k\right) = G(k) = \text{DFT}_L\{g(n)\}$$

where $G(k)$ is the L-point DFT of $g(n)$.

Types I and II

Suppose the FIR filter $h(n)$ is either a Type I or a Type II FIR filter. Then we have from above

$$H^f(\omega) = A(\omega)\, e^{-jM\omega}$$

or

$$A(\omega) = H^f(\omega)\, e^{jM\omega}.$$

Samples of the real-valued amplitude $A(\omega)$ can be obtained from samples of the function $H^f(\omega)$ as:

$$A\left(\frac{2\pi}{L}k\right) = H\left(\frac{2\pi}{L}k\right) e^{jM\frac{2\pi}{L}k} = G(k) \cdot W_L^{Mk}.$$

Therefore, the samples of the real-valued amplitude function can be obtained by zero-padding $h(n)$, taking the DFT, and multiplying by the complex exponential. This can be written as:

$$\boxed{A\left(\frac{2\pi}{L}k\right) = \mathsf{DFT}_L\{[h(n),\ 0_{L-N}]\} \cdot W_L^{Mk}.}$$

Types III and IV

For Type III and Type IV FIR filters, we have

$$H^f(\omega) = j\, e^{-jM\omega}\, A(\omega)$$

or

$$A(\omega) = -j\, H^f(\omega)\, e^{jM\omega}.$$

Therefore, samples of the real-valued amplitude $A(\omega)$ can be obtained from samples of the function $H^f(\omega)$ as:

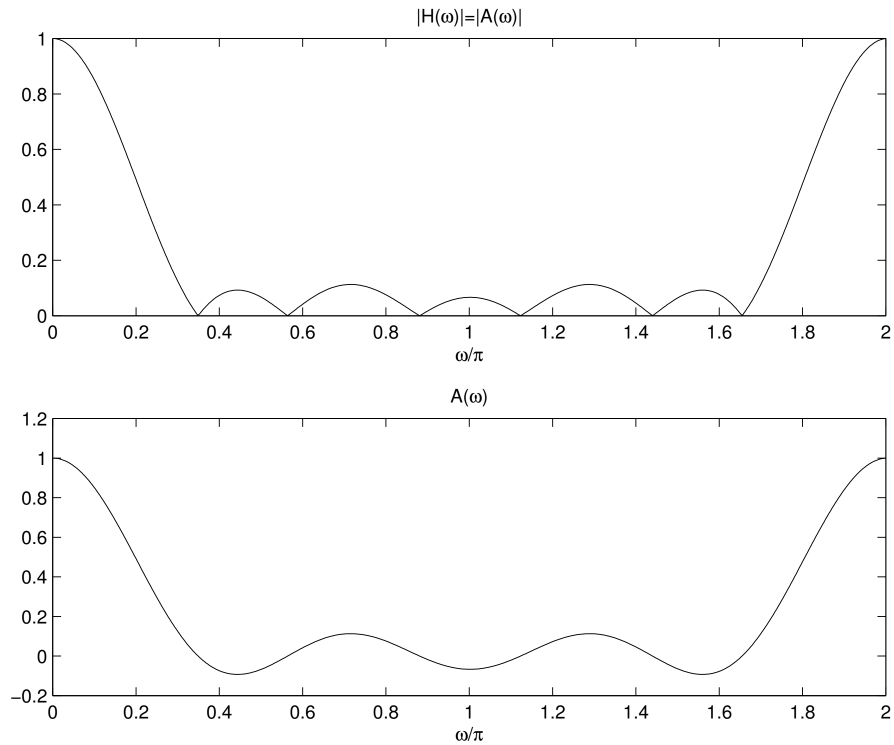
$$A\left(\frac{2\pi}{L}k\right) = -j\, H\left(\frac{2\pi}{L}k\right)\, e^{jM\frac{2\pi}{L}k} = -j\, G(k) \cdot W_L^{Mk}.$$

Therefore, the samples of the real-valued amplitude function can be obtained by zero-padding $h(n)$, taking the DFT, and multiplying by the complex exponential.

$$\boxed{A\left(\frac{2\pi}{L}k\right) = -j \cdot \mathsf{DFT}_L\{[h(n),\ 0_{L-N}]\} \cdot W_L^{Mk}.}$$

EXAMPLE: EVALUATING THE AMP RESP (TYPE I)

In this example, the filter is a Type I FIR filter of length 7. An accurate plot of $A(\omega)$ can be obtained with zero padding.



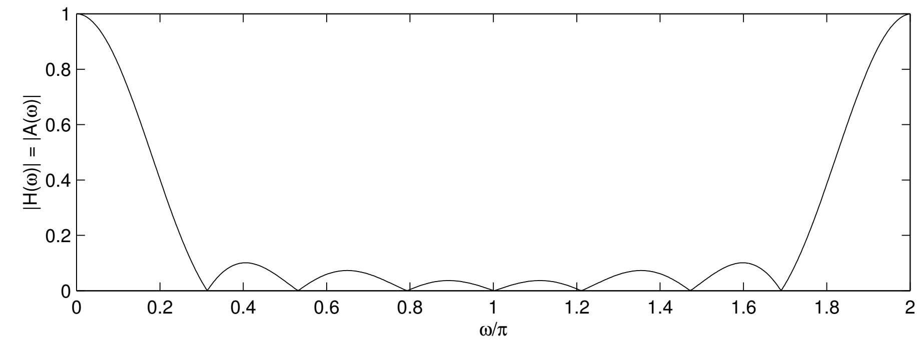
The following Matlab code fragment for the plot of $A(\omega)$ for a Type I FIR filter.

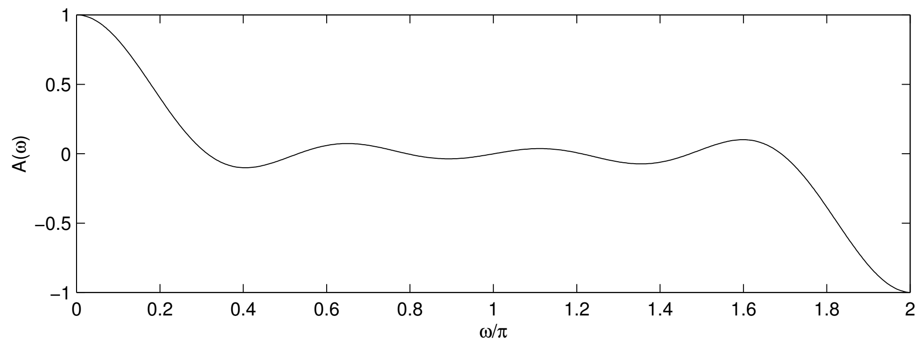
```
h = [3 4 5 6 5 4 3]/30;
N = 7;
M = (N-1)/2;
L = 512;
H = fft([h zeros(1,L-N)]);
k = 0:L-1;
W = exp(j*2*pi/L);
A = H .* W.^(M*k);
A = real(A);

figure(1)
w = [0:L-1]*2*pi/L;
```

```
subplot(2,1,1)
plot(w/pi,abs(H))
ylabel('|H(\omega)| = |A(\omega)|')
xlabel('\omega/\pi')
subplot(2,1,2)
plot(w/pi,A)
ylabel('A(\omega)')
xlabel('\omega/\pi')
print -deps type1
```

The command `A = real(A)` removes the imaginary part which is equal to zero to within computer precision. Without this command, Matlab takes `A` to be a complex vector and the following plot command will not be right.

Observe the symmetry of $A(\omega)$ due to $h(n)$ being real-valued. Because of this symmetry, $A(\omega)$ is usually plotted for $0 \le \omega \le \pi$ only.





The following Matlab code fragment produces a plot of $A(\omega)$ for a Type II FIR filter.

```
h = [3 5 6 7 7 6 5 3]/42;
N = 8;
M = (N-1)/2;
L = 512;
H = fft([h zeros(1,L-N)]);
k = 0:L-1;
W = exp(j*2*pi/L);
A = H .* W.^(M*k);
A = real(A);

figure(1)
w = [0:L-1]*2*pi/L;
subplot(2,1,1)
plot(w/pi,abs(H))
ylabel('|H(\omega)| = |A(\omega)|')
xlabel('\omega/\pi')
```

```
subplot(2,1,2)
plot(w/pi,A)
ylabel('A(\omega)')
xlabel('\omega/\pi')
print -deps type2
```

The imaginary part of the amplitude is zero. Notice that $A(\pi) = 0$. In fact this will always be the case for a Type II FIR filter.

An exercise for the student: Describe how to obtain samples of $A(\omega)$ for Type III and Type IV FIR filters. Modify the Matlab code above for these types. Do you notice that $A(\omega) = 0$ always for special values of ω?

ZERO LOCATIONS OF LINEAR-PHASE FILTERS

Sec 4.4.4 in Mitra

The zeros of the transfer function $H(z)$ of a linear-phase filter lie in specific configurations.

We can write the symmetry condition

$$h(n) = h(N - 1 - n)$$

in the Z domain. Taking the Z-transform of both sides gives

$$H(z) = z^{-(N-1)}\, H(1/z). \tag{25}$$

Recall that we are assuming that $h(n)$ is real-valued. If z_o is a zero of $H(z)$,

$$H(z_o) = 0,$$

then

$$H(z_o^*) = 0.$$

(Because the roots of a polynomial with real coefficients exist in complex-conjugate pairs.)

Using the symmetry condition (25), it follows that

$$H(z_o) = z_o^{-(N-1)}\, H(1/z_o) = 0$$

and

$$H(z_o^*) = (z_o^*)^{-(N-1)}\, H(1/z_o^*) = 0$$

or

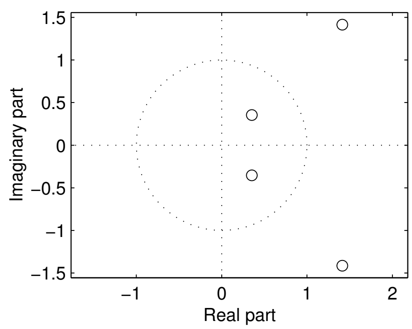
$$H(1/z_o) = H(1/z_o^*) = 0.$$

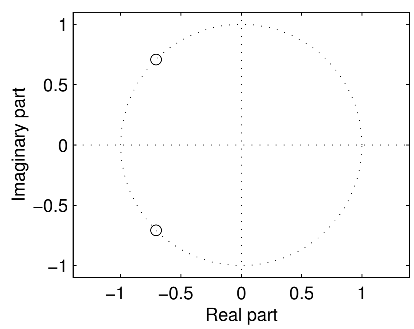
If z_o is a zero of a (real-valued) linear-phase filter, then so are z_o^*, $1/z_o$, $1/z_o^*$.

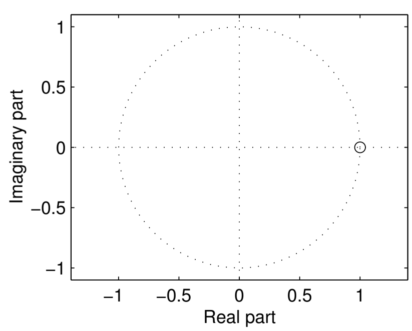
It follows that

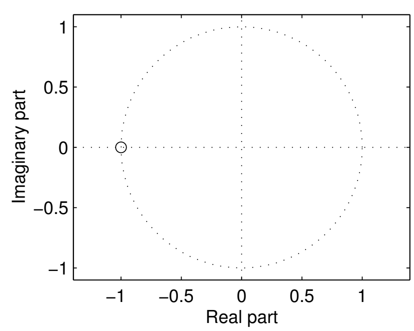
1. generic zeros of a linear-phase filter exist in sets of 4.
2. zeros on the unit circle ($z_o = e^{j\omega_o}$) exist in sets of 2. ($z_0 \neq \pm 1$)
3. zeros on the real line ($z_o = a$) exist in sets of 2. ($z_0 \neq \pm 1$)
4. zeros at 1 and -1 do not imply the existence of zeros at other specific points.

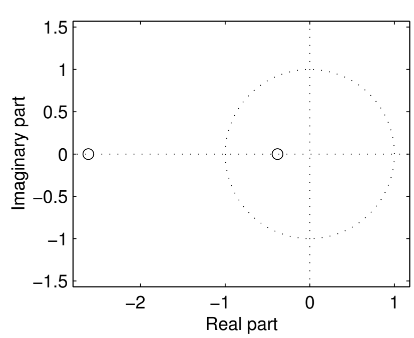
Examples of zero sets:











ZERO LOCATIONS: AUTOMATIC ZEROS

The frequency response $H^f(\omega)$ of a Type II FIR filter always has a zero at $\omega = \pi$:

$$h(n) = [h_0,\ h_1,\ h_2,\ h_2,\ h_1,\ h_0]$$

$$H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + h_2 z^{-3} + h_1 z^{-4} + h_0 z^{-5}$$

$$H(-1) = h_0 - h_1 + h_2 - h_2 + h_1 - h_0 = 0$$

$$H^f(\pi) = H(e^{j\pi}) = H(-1) = 0$$

$H^f(\pi) = 0$ always for Type II filters.

Similarly, we can derive the following rules for Type III and Type IV FIR filters.

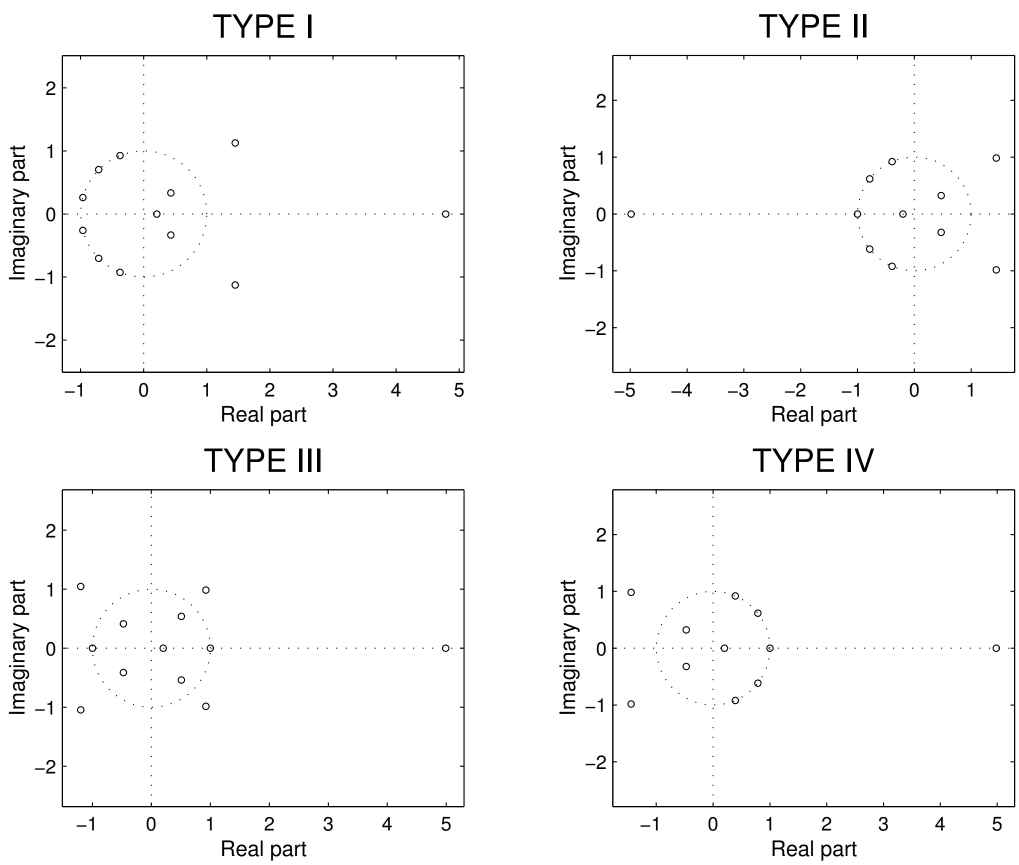
$H^f(0) = H^f(\pi) = 0$ always for Type III filters.

$H^f(0) = 0$ always for Type IV filters.

The automatic zeros can also be derived using the characteristics of the amplitude response $A(\omega)$ seen earlier.

Type	automatic zeros
I	—
II	$\omega = \pi$
III	$\omega = 0, \pi$
IV	$\omega = 0$

The Matlab command `zplane` can be used to plot the zero locations of FIR filters.



Note that the zero locations satisfy the properties noted previously.

One approach to the design of FIR filters is to ask that $A(\omega)$ pass through a specified set of values. If the number of specified interpolation points is the same as the number of filter parameters, then the filter is totally determined by the interpolation conditions, and the filter can be found by solving a system of linear equations.

When the interpolation points are equally spaced between 0 and 2π, then this interpolation problem can be solved very efficiently using the DFT.

To derive the DFT solution to the interpolation problem, recall the formula relating the samples of the frequency response to the DFT. In the case we are interested here, the number of samples is to be the *same* as the length of the filter ($L = N$).

$$H\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} h(n) e^{-j\frac{2\pi}{N}nk}$$
$$= \mathrm{DFT}_N\{h(n)\}.$$

Types I and II

Recall the relation between $A(\omega)$ and $H^f(\omega)$ for a Type I and II filter, to obtain

$$A\left(\frac{2\pi}{N}k\right) = H\left(\frac{2\pi}{N}k\right) \cdot e^{jM\frac{2\pi}{N}k}$$
$$= \mathrm{DFT}_N\{h(n)\} \cdot W_N^{Mk}$$

Now we can related the N-point DFT of $h(n)$ to the samples of $A(\omega)$:

$$\mathrm{DFT}_N\{h(n)\} = A\left(\frac{2\pi}{N}k\right) \cdot W_N^{-Mk}.$$

Finally, we can solve for the filter coefficients $h(n)$.

$$\boxed{h(n) = \mathsf{DFT}_N^{-1}\left\{A\left(\frac{2\pi}{N}k\right)\cdot W_N^{-Mk}\right\}.}$$

Therefore, if the values $A\left(\frac{2\pi}{N}k\right)$ are specified, we can then obtain the filter coefficients $h(n)$ that satisfies the interpolation conditions by using the inverse DFT. It is important to note however, that the specified values $A\left(\frac{2\pi}{N}k\right)$ must posses the appropriate symmetry in order for the result of the inverse DFT to be a real Type I or II FIR filter.

Types III and IV

For Type III and IV filters, we have

$$\begin{aligned} A\left(\frac{2\pi}{N}k\right) &= -j\cdot H\left(\frac{2\pi}{N}k\right)\cdot e^{jM\frac{2\pi}{N}k} \\ &= -j\cdot \mathsf{DFT}_N\{h(n)\}\cdot W_N^{Mk} \end{aligned}$$

Then we can related the N-point DFT of $h(n)$ to the samples of $A(\omega)$:

$$\mathsf{DFT}_N\{h(n)\} = j\cdot A\left(\frac{2\pi}{N}k\right)\cdot W_N^{-Mk}.$$

Solving for the filter coefficients $h(n)$ gives:

$$\boxed{h(n) = \mathsf{DFT}_N^{-1}\left\{j\cdot A\left(\frac{2\pi}{N}k\right)\cdot W_N^{-Mk}\right\}.}$$

EXAMPLE: DFT-INTERPOLATION (TYPE I)

The following Matlab code fragment illustrates how to use this approach to design a length 11 Type I FIR filter for which

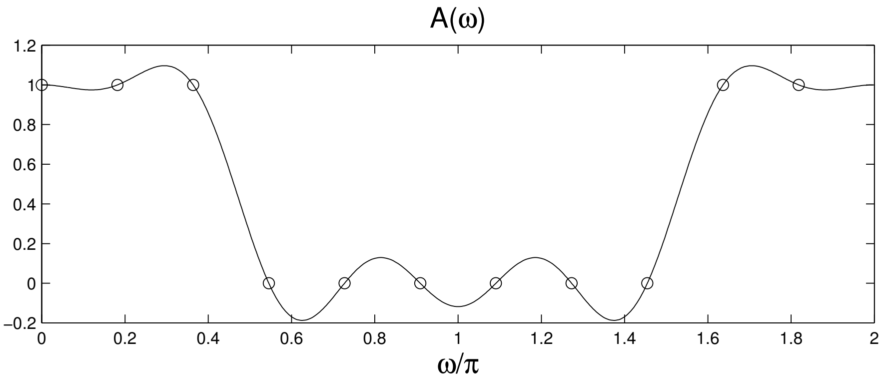
$$A\left(\frac{2\pi}{N}k\right) = (1,1,1,0,0,0,0,0,0,1,1) \qquad 0 \le k \le N-1 \qquad N = 11.$$

```
>> N = 11;
>> M = (N-1)/2;
>> Ak = [1 1 1 0 0 0 0 0 0 1 1];   % samples of A(w)
>> k = 0:N-1;
>> W = exp(j*2*pi/N);
>> h = ifft(Ak.*W.^(-M*k));
>> h'

ans =

   0.0694 - 0.0000i
  -0.0540 - 0.0000i
  -0.1094 + 0.0000i
   0.0474 + 0.0000i
   0.3194 + 0.0000i
   0.4545 + 0.0000i
   0.3194 + 0.0000i
   0.0474 - 0.0000i
  -0.1094 - 0.0000i
  -0.0540 - 0.0000i
   0.0694 - 0.0000i
```

Observe that the filter coefficients `h` are real and symmetric; that a Type I filter is obtained as desired. The plot of $A(\omega)$ for this filter illustrates the interpolation points.

```
L = 512;
H = fft([h zeros(1,L-N)]);
W = exp(j*2*pi/L);
k = 0:L-1;
A = H .* W.^(M*k);
A = real(A);
```

```
w = k*2*pi/L;
plot(w/pi,A,2*[0:N-1]/N,Ak,'o')
xlabel('\omega/\pi')
title('A(\omega)')
```



An exercise for the student: develop this DFT-based interpolation approach for Type II, III, and IV FIR filters. Modify the Matlab code above for each case.

SUMMARY: IMPULSE AND AMP RESPONSE

For an N-point linear-phase FIR filter $h(n)$, we summarize:

1. The formulas for evaluating the amplitude response $A(\omega)$ at L equally spaced points from 0 to 2π ($L \geq N$).

2. The formulas for the DFT-based interpolation design of $h(n)$.

TYPE I and II:

$$A\left(\frac{2\pi}{L}k\right) = \mathsf{DFT}_L\{[h(n),\ 0_{L-N}]\} \cdot W_L^{Mk}$$

$$h(n) = \mathsf{DFT}_N^{-1}\left\{A\left(\frac{2\pi}{N}k\right) \cdot W_N^{-Mk}\right\}$$

TYPE III and IV:

$$A\left(\frac{2\pi}{L}k\right) = -j \cdot \mathsf{DFT}_L\{[h(n),\ 0_{L-N}]\} \cdot W_L^{Mk}$$

$$h(n) = \mathsf{DFT}_N^{-1}\left\{j \cdot A\left(\frac{2\pi}{N}k\right) \cdot W_N^{-Mk}\right\}$$

DESIGN OF FIR FILTERS BY GENERAL INTERPOLATION

If the desired interpolation points are not uniformly spaced between 0 and π then we can not use the DFT. We must take a different approach. Recall that for a Type I FIR filter,

$$A(\omega) = h(M) + 2\sum_{n=0}^{M-1} h(n)\cos\left((M-n)\omega\right).$$

For convenience, it is common to write this as

$$A(\omega) = \sum_{n=0}^{M} a(n)\cos\left(n\omega\right)$$

where

$$h(M) = a(0), \qquad h(n) = a(M-n)/2 \quad 1 \le n \le N-1.$$

Note that there are $M+1$ parameters. Suppose it is desired that $A(\omega)$ interpolates a set of specified values:

$$A(\omega_k) = A_k, \qquad 0 \le k \le M.$$

To obtain a Type I FIR filter satisfying these interpolation equations, one can set up a linear system of equations.

$$\sum_{n=0}^{M} a(n)\cos\left(n\,\omega_k\right) = A_k, \qquad 0 \le k \le M.$$

In matrix form, we have

$$\begin{bmatrix} 1 & \cos(w_0) & \cos(2w_0) & \cdots & \cos(Mw_0) \\ 1 & \cos(w_1) & \cos(2w_1) & \cdots & \cos(Mw_1) \\ \vdots & & & & \vdots \\ 1 & \cos(w_M) & \cos(2w_M) & \cdots & \cos(Mw_M) \end{bmatrix} \cdot \begin{bmatrix} a(0) \\ a(1) \\ \vdots \\ a(M) \end{bmatrix} = \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_M \end{bmatrix}$$

Once $a(n)$ is found, the filter $h(n)$ is formed as

$$\{h(n)\} = \frac{1}{2}\cdot\{a(M), a(M-1), \ldots, a(1), 2\,a(0), a(1), \ldots, a(M-1), a(M)\}.$$

EXAMPLE

In the following example, we design a length 19 Type I FIR. Then $M = 9$ and we have 10 parameters. We can therefore have 10 interpolation equations. We choose:

$$A(\omega_k) = 1, \qquad \omega_k = \{0, 0.1\,\pi, 0.2\,\pi, 0.3\,\pi\}, \quad 0 \le k \le 3 \tag{26}$$

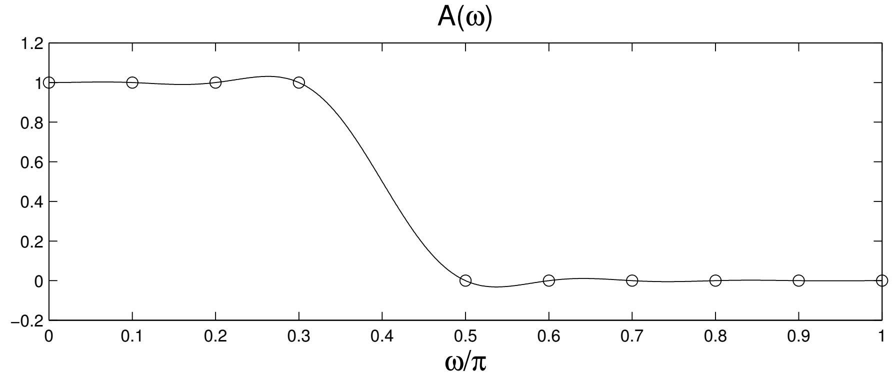
$$A(\omega_k) = 0, \qquad \omega_k = \{0.5\,\pi, 0.6\,\pi, 0.7\,\pi, 0.8\,\pi, 0.9\,\pi, 1.0\,\pi\}, \quad 4 \le k \le 9. \tag{27}$$

To solve this interpolation problem in Matlab, note that the matrix can be generated by a single multiplication of a column vector and a row vector. This is done with the command

```
C = cos(wk*[0:M]);
```

where `wk` is a column vector containing the frequency points. To solve the linear system of equations, we can use the Matlab backslash command.

```
N = 19;
M = (N-1)/2;
wk = [0 .1 .2 .3 .5 .6 .7 .8 .9 1]'*pi;
Ak = [1 1 1 1 0 0 0 0 0 0]';
C = cos(wk*[0:M]);
a = C\Ak;
h = (1/2)*[a([M:-1:1]+1); 2*a([0]+1); a([1:M]+1)];

[A,w] = firamp(h,1);
plot(w/pi,A,wk/pi,Ak,'o')
title('A(\omega)')
xlabel('\omega/\pi')
```



The general interpolation problem is much more flexible than the uniform interpolation problem that the DFT solves. For example, by leaving a gap between the pass-band and stop-band as in this example, the ripple near the band edge is reduced (but the transition between the pass- and stop-bands is not as sharp). The general interpolation problem also arises as a subproblem in the design of optimal minimax (or Chebyshev) FIR filters.