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DFT DEFINITION

The DFT of an N-point signal

$$x[n], \quad n \in \mathbb{Z}_N := \{0, 1, \ldots, N-1\}$$

is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n] \, W_N^{-kn}, \qquad k \in \mathbb{Z}_N \tag{1}$$

where

$$W_N = \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}} = \cos\Big(\frac{2\pi}{N}\Big) + \mathrm{j}\sin\Big(\frac{2\pi}{N}\Big)$$

is the principal N-th root of unity.

The original sequence $x[n]$ can be retrieved by the inverse discrete Fourier transform (IDFT)

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, W_N^{kn}, \qquad n \in \mathbb{Z}_N.$$

THE MOD NOTATION

The notation $\langle k \rangle_N$ denotes the remainder r when k is divided by N. This is also denoted as $k \mod N$. For example

$$\langle 3 \rangle_4 = 3 \qquad \text{and} \qquad \langle 6 \rangle_4 = 2.$$

The following table shows $\langle k \rangle_4$ for values of k from 0 to 7.

k	$\langle k \rangle_4$
0	0
1	1
2	2
3	3
4	0
5	1
6	2
7	3
$\vdots$	$\vdots$

The notation $\langle k \rangle_N$ is also defined for negative integers k. We choose the remainder r so that it is between 0 and $N-1$.

$$0 \le \langle k \rangle_N \le N-1$$

For example,

$$-3 = (-1)(4) + 1 \qquad \text{so} \qquad \langle -3 \rangle_4 = 1$$

and

$$-6 = (-2)(4) + 2 \qquad \text{so} \qquad \langle -6 \rangle_4 = 2$$

THE MOD NOTATION

The following table shows $\langle k \rangle_4$ for values of k from -8 to 7.

k	$\langle k \rangle_4$
$\vdots$	$\vdots$
-8	0
-7	1
-6	2
-5	3
-4	0
-3	1
-2	2
-1	3
0	0
1	1
2	2
3	3
4	0
5	1
6	2
7	3
$\vdots$	$\vdots$

In MATLAB, we can generate $\langle k \rangle_N$ using the command `mod`, as illustrated by the code fragment:

```
>> k=-8:7;
>> [k', mod(k,4)']

ans =

    -8     0
    -7     1
    -6     2
    -5     3
    -4     0
    -3     1
    -2     2
    -1     3
     0     0
     1     1
     2     2
     3     3
     4     0
     5     1
     6     2
     7     3
```

Evidently $\langle k \rangle_N$ is a periodic function of k with period N:

$$\langle k + N \rangle_N = \langle k \rangle_N$$

PERIODICITY OF W_N

Our definition of W_N is

$$W_N := \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}}.$$

If we write k as $k = l \cdot N + r$ with $r \in \mathbb{Z}_N$ then we have $\langle k \rangle_N = r$ and

$$\begin{aligned} W_N^k &= \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}(l \cdot N + r)} \\ &= \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}(l \cdot N)} \cdot \mathrm{e}^{\mathrm{j}\frac{2\pi}{N} r} \\ &= 1 \cdot W_N^r \\ &= W_N^{\langle k \rangle_N}. \end{aligned}$$

Therefore W_N^k is a periodic function of k with a period of N. The identity

$$\boxed{W_N^k = W_N^{\langle k \rangle_N}}$$

is useful when working with the DFT.

A related identity is

$$\boxed{W_N^{m \cdot k} = W_N^{m \cdot \langle k \rangle_N}}$$

which is derived as

$$\begin{aligned} W_N^{m \cdot k} &= \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}(m)(l \cdot N + r)} \\ &= \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}(m \cdot l \cdot N)} \cdot \mathrm{e}^{\mathrm{j}\frac{2\pi}{N} m \cdot r} \\ &= 1 \cdot W_N^{m \cdot r} \\ &= W_N^{m \cdot \langle k \rangle_N}. \end{aligned}$$

A USEFUL IDENTITY

First, recall the geometric summation formula:

$$\sum_{n=0}^{N-1} a^n = \begin{cases} \frac{1-a^N}{1-a} & a \neq 1 \\ N & a = 1. \end{cases}$$

If $a = W_N^k$, then we get:

$$\sum_{n=0}^{N-1} W_N^{nk} = \begin{cases} \frac{1-W_N^{Nk}}{1-W_N^k} & W_N^k \neq 1 \\ N & W_N^k = 1. \end{cases}$$

Because $W_N^{Nk} = 1$, this simplifies to

$$\sum_{n=0}^{N-1} W_N^{nk} = \begin{cases} 0 & W_N^k \neq 1 \\ N & W_N^k = 1. \end{cases}$$

Note that $W_N^k = 1$ if $k = 0, \pm N, \pm 2N, \ldots$. In other words $W_N^k = 1$ if $\langle k \rangle_N = 0$. Similarly $W_N^k \neq 1$ if $\langle k \rangle_N \neq 0$. Therefore we can simplify the summation further to obtain:

$$\sum_{n=0}^{N-1} W_N^{nk} = \begin{cases} 0 & \langle k \rangle_N \neq 0 \\ N & \langle k \rangle_N = 0. \end{cases}$$

Using the discrete-time delta function $\delta[n]$, defined as

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0, \end{cases}$$

we finally get the sought identity in the simple form:

$$\boxed{\sum_{n=0}^{N-1} W_N^{nk} = N \cdot \delta[\langle k \rangle_N].}$$

INVERSE DFT PROOF

With the formula

$$\sum_{n=0}^{N-1} W_N^{nk} = N \cdot \delta[\langle k \rangle_N]$$

we are ready to verify the the inverse DFT formula.

To verify the inversion formula, we substitute the DFT into the inverse DFT formula:

$$\begin{aligned}
x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, W_N^{kn} \\
x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_{l=0}^{N-1} x[l]\, W_N^{-kl} \right) W_N^{kn}, \\
&= \frac{1}{N} \sum_{l=0}^{N-1} x[l] \cdot \sum_{k=0}^{N-1} W_N^{k(n-l)}, \\
&= \frac{1}{N} \sum_{l=0}^{N-1} x[l] \cdot N \cdot \delta[\langle n-l \rangle_N], \\
&= x[\langle n \rangle_N] \\
&= x[n] \qquad \text{provided } 0 \le n \le N-1.
\end{aligned}$$

CIRCULAR SHIFTING

Given an N-point signal $\{x[n],\ n \in \mathbb{Z}_N\}$, the signal

$$g[n] := x[\langle n - m \rangle_N]$$

represents a circular shift of $x[n]$ by m samples to the right. For example, if

$$g[n] := x[\langle n - 1 \rangle_N]$$

then

$$\begin{aligned} g[0] &= x[\langle -1 \rangle_N] = x[N-1] \\ g[1] &= x[\langle 0 \rangle_N] = x[0] \\ g[2] &= x[\langle 1 \rangle_N] = x[1] \\ &\vdots \\ g[N-1] &= x[\langle N-2 \rangle_N] = x[N-2] \end{aligned}$$

For example, if $x[n]$ is the 4-point signal

$$x[n] = (1, 3, 5, 2)$$

then

$$x[\langle n - 1 \rangle_N] = (2, 1, 3, 5).$$

$\boxed{x[\langle n - m \rangle_N] \text{ represents a } \textit{circular} \text{ shift by } m \text{ samples.}}$

CIRCULAR SHIFTING

The following MATLAB code fragment illustrates the circular shift.

```
>> g = [1 3 5 2]
g =
     1     3     5     2


>> n = 0:3
n =
     0     1     2     3


>> mod(n-1,4)
     3     0     1     2


>> g(mod(n-1,4)+1)      % Add 1 to account for MATLAB indexing
     2     1     3     5
```

Because indexing in MATLAB begins with 1 rather than with 0, it is necessary to add 1 to the index vector.

CIRCULAR CONVOLUTION

The usual convolution (or linear convolution) is defined as

$$x[n] * g[n] := \sum_{m=0}^{N-1} x[m]g[n-m].$$

But when $x[n]$ and $g[n]$ are defined as N-point signals

$$x[n], \quad n \in \mathbb{Z}_N \qquad \text{and} \qquad g[n], \quad n \in \mathbb{Z}_N$$

then the term $g[n-m]$ will fall outside the range. The definition of *circular* convolution evaluates the index modulo N:

$$\boxed{x[n] \circledast g[n] := \sum_{m=0}^{N-1} x[m]g[\langle n-m \rangle_N].}$$

The following MATLAB function illustrates how the circular convolution can be computed.

```
function y = cconv(x,g)
% y = cconv(x,g)

N = length(x);
n = 0:N-1;
y = zeros(size(x));
for m = 0:N-1
   y = y + x(m+1)*g(mod(n-m,N)+1);
end
```

For example:

```
>> x = [2 4 3 1 5];
>> g = [7 3 5 2 4];
>> y = cconv(x,g)
y =
    56    73    57    60    69
```

CIRCULAR TIME-REVERSAL

Given an N-point signal $\{x[n],\ n \in \mathbb{Z}_N\}$, the signal

$$g[n] := x[\langle -n \rangle_N]$$

is given by

$$g[n] = \begin{cases} x[0] & n = 0 \\ x[N-n] & 1 \le n \le N-1. \end{cases}$$

as shown by:

$$\begin{aligned} g[0] &= x[\langle 0 \rangle_N] = x[0] \\ g[1] &= x[\langle -1 \rangle_N] = x[N-1] \\ g[2] &= x[\langle -2 \rangle_N] = x[N-2] \\ &\vdots \\ g[N-1] &= x[\langle 1-N \rangle_N] = x[1]. \end{aligned}$$

For example, if $x[n]$ is the 4-point signal

$$x[n] = (1, 3, 5, 2)$$

then

$$x[\langle -n \rangle_N] = (1, 2, 5, 3).$$

CIRCULAR TIME-REVERSAL

The following MATLAB code fragment illustrates circular time-reversal for a 4-point signal.

```
>> g = [1 3 5 2]

g =

     1     3     5     2

>> N = 4;
>> n = 0:N-1;
>> g(mod(-n,N)+1)

ans =

     1     2     5     3
```

CIRCULAR SYMMETRY

A discrete-time signal $x[n]$ is symmetric if

$$x[n] = x[-n].$$

But if $x[n]$ is an N-point signal $\{x[n],\ n \in \mathbb{Z}_N\}$, then the sample $x[-n]$ falls outside the range. The definition of *circular* symmetry evaluates the index modulo N. An N-point signal $x[n]$ is circularly symmetric if

$$x[n] = x[\langle -n \rangle_N].$$

For example, the sequences

$$x[n] = (5, 3, 4, 2, 2, 4, 3)$$
$$v[n] = (5, 3, 4, 2, 7, 2, 4, 3)$$

are both circularly symmetric.

PERIODICITY PROPERTY OF THE DFT

Given the N-point signal $\{x[n],\ n \in \mathbb{Z}_N\}$, we defined the DFT coefficients $X[k]$ for $0 \le k \le N-1$. But if k lies *outside* the range $0, \dots, N-1$, then

$$\boxed{X[k] = X[\langle k \rangle_N].}$$

To derive this equation, write k as $k = l \cdot N + r$ with $r \in \mathbb{Z}_N$. Then $\langle k \rangle_N = r$, and

$$\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n]\, W_N^{-n \cdot (l \cdot N + r)} \\
&= \sum_{n=0}^{N-1} x[n]\, W_N^{-n\, l\, N} \cdot W_N^{-nr} \\
&= \sum_{n=0}^{N-1} x[n]\, W_N^{-nr} \\
&= X[r] \\
&= X[\langle k \rangle_N].
\end{aligned}$$

If the DFT formula is evaluated for k outside the range $k \in \mathbb{Z}_N$, then one finds that $X[k]$ is periodic with period N.

PERIODICITY PROPERTY OF THE DFT

Likewise, given the N-point DFT vector $\{X[k],\ k \in \mathbb{Z}_N\}$, we defined the *Inverse* DFT samples $x[n]$ for $n \in \mathbb{Z}_N$. But if n lies *outside* the range $0, \dots, N-1$, then

$$\boxed{x[n] = x[\langle n \rangle_N].}$$

To derive this equation, write n as $n = l \cdot N + r$ with $r \in \mathbb{Z}_N$. Then $\langle n \rangle_N = r$, and

$$\begin{aligned}
x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, W_N^{k\cdot(l\cdot N + r)} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, W_N^{k\,l\,N} \cdot W_N^{kr} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, W_N^{kr} \\
&= x[r] \\
&= x[\langle n \rangle_N].
\end{aligned}$$

If the Inverse DFT formula is evaluated for n outside the range $n \in \mathbb{Z}_N$, then one finds that $x[n]$ is periodic with period N.

CIRCULAR SHIFT PROPERTY OF THE DFT

If

$$G[k] := W_N^{-mk} \cdot X[k]$$

then

$$g[n] = x[\langle n - m \rangle_N].$$

Derivation:

Begin with the Inverse DFT.

$$\begin{aligned} g[n] &= \frac{1}{N} \sum_{k=0}^{N-1} G[k] \, W_N^{nk} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} W_N^{-mk} \, X[k] \, W_N^{nk} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, W_N^{k(n-m)} \\ &= x[n - m] \\ &= x[\langle n - m \rangle_N]. \end{aligned}$$

CIRCULAR SHIFT PROPERTY OF THE DFT

The following MATLAB code fragment illustrates the circular shift property with a shift of 2 samples. property.

```
>> x = [3 1 5 2 4]';
>>
>> N = length(x);   % N: data length
>> W = exp(2*pi/N*sqrt(-1));
>> X = fft(x);
>> k = [0:N-1]';    % frequency index
>> m = 2;           % shift
>> G = W.^(-m*k) .* X;
>> g = ifft(G)

g =

   2.0000 - 0.0000i
   4.0000 + 0.0000i
   3.0000 - 0.0000i
   1.0000 - 0.0000i
   5.0000 - 0.0000i
```

MODULATION PROPERTY (circular shift in frequency)

If

$$g[n] := W_N^{mn} \cdot x[n]$$

then

$$G[k] = X[\langle k - m \rangle_N].$$

Derivation:

Begin with the DFT.

$$\begin{aligned} G[k] &= \sum_{n=0}^{N-1} g[n] \, W_N^{-nk} \\ &= \sum_{n=0}^{N-1} W_N^{mn} \, x[n] \, W_N^{-nk} \\ &= \sum_{n=0}^{N-1} x[n] \, W_N^{-n(k-m)} \\ &= X[k-m] \\ &= X[\langle k - m \rangle_N]. \end{aligned}$$

If

$$y[n] := \sum_{m=0}^{N-1} x[m] g[\langle n-m \rangle_N]$$

then

$$Y[k] = X[k] \cdot G[k].$$

Derivation:

$$\begin{aligned}
Y[k] &= \sum_{n=0}^{N-1} y[n] W_N^{-nk} \\
&= \sum_{n=0}^{N-1} \cdot \sum_{m=0}^{N-1} x[m] g[\langle n-m \rangle_N] W_N^{-nk} \\
&= \sum_{m=0}^{N-1} x[m] \cdot \sum_{n=0}^{N-1} g[\langle n-m \rangle_N] W_N^{-nk} \\
&= \sum_{m=0}^{N-1} x[m] W_N^{-mk} G[k] \\
&= G[k] \cdot \sum_{m=0}^{N-1} x[m] W_N^{-mk} \\
&= G[k] \cdot X[k]
\end{aligned}$$

CIRCULAR CONVOLUTION PROPERTY OF THE DFT

The following MATLAB code fragment illustrates the DFT convolution theorem.

```
>> x = [2 4 3 1 5];
>> g = [7 3 5 2 4];
>> cconv(x,g)

ans = [ 56 73 57 60 69 ]

>> ifft(fft(x).*fft(g))'

ans =

  56.0000 + 0.0000i
  73.0000 - 0.0000i
  57.0000 + 0.0000i
  60.0000 - 0.0000i
  69.0000 - 0.0000i
```

$$\sum_{n=0}^{N-1} x[n] \cdot g^*[n] = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X[k] \cdot G^*[k]$$

Derivation:

$$\begin{aligned}
\sum_{n=0}^{N-1} x[n] \cdot g^*[n] &= \sum_{n=0}^{N-1} x[n] \cdot \left(\frac{1}{N} \sum_{k=0}^{N-1} G[k] W_N^{nk} \right)^* \\
&= \frac{1}{N} \sum_{n=0}^{N-1} x[n] \cdot \sum_{k=0}^{N-1} G^*[k] W_N^{-nk} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} G^*[k] \cdot \sum_{n=0}^{N-1} x[n] W_N^{-nk} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cdot G^*[k]
\end{aligned}$$

PARSEVAL'S THEOREM FOR THE DFT

The following MATLAB code fragment illustrates Parseval's theorem.

```
>> N = 4;
>> x = randn(1,N) + i*randn(1,N)

x =
   0.7812 - 1.1878i   0.5690 - 2.2023i  -0.8217 + 0.9863i  -0.2656 - 0.5186i

>> g = randn(1,N) + i*randn(1,N)

g =
   0.3274 - 0.9471i   0.2341 - 0.3744i   0.0215 - 1.1859i  -1.0039 - 1.0559i

>> X = fft(x);
>> G = fft(g);
>> sum(x.*conj(g))

ans =
   1.9655 - 0.6644i

>> sum(X.*conj(G))/N

ans =
   1.9655 - 0.6644i
```

TIME-REVERSAL PROPERTY OF THE DFT

If

$$g[n] := x[\langle -n \rangle_N]$$

then

$$G[k] = X[\langle -k \rangle_N].$$

Derivation:

$$\begin{aligned}
G[k] &= \sum_{n=0}^{N-1} x[\langle -n \rangle_N] W_N^{-nk} \\
&= \sum_{m=0}^{N-1} x[m] W_N^{-\langle -m \rangle_N k} \\
&= \sum_{m=0}^{N-1} x[m] W_N^{mk} \\
&= X[-k] \\
&= X[\langle -k \rangle_N]
\end{aligned}$$

where we used the change of variables $m = \langle -n \rangle_N$ (in which case $n = \langle -m \rangle_N$ for $0 \leq n \leq N-1$).

TIME-REVERSAL PROPERTY OF THE DFT

The following MATLAB code fragment illustrates the time-reversal property.

```
>> N = 5;
>> n = 0:N-1;
>> x = [3 6 2 4 7]';
>> X = fft(x);
>> G = X(mod(-n,N)+1);
>> g = ifft(G)

g =

   3.0000 - 0.0000i
   7.0000 + 0.0000i
   4.0000 + 0.0000i
   2.0000
   6.0000 - 0.0000i
```

COMPLEX CONJUGATION PROPERTY OF THE DFT

If

$$g[n] := x^*[n]$$

then

$$G[k] := X^*[\langle -k \rangle_N]$$

Derivation:

$$\begin{aligned} G[k] &= \sum_{n=0}^{N-1} x^*[n] W_N^{-nk} \\ &= \left(\sum_{n=0}^{N-1} x[n] W_N^{nk} \right)^* \\ &= X^*[-k] \\ &= X^*[\langle -k \rangle_N] \end{aligned}$$

COMPLEX CONJUGATION PROPERTY OF THE DFT

The following MATLAB code fragment illustrates the complex conjugation property.

```
>> N = 5;
>> n = 0:N-1;
>> i = sqrt(-1);
>> x = [3+2*i 5-3*i 6+i 4-7*i 8+9*i].'

x =

   3.0000 + 2.0000i
   5.0000 - 3.0000i
   6.0000 + 1.0000i
   4.0000 - 7.0000i
   8.0000 + 9.0000i

>> X = fft(x);
>> G = conj(X(mod(-n,N)+1));
>> g = ifft(G)

g =

   3.0000 - 2.0000i
   5.0000 + 3.0000i
   6.0000 - 1.0000i
   4.0000 + 7.0000i
   8.0000 - 9.0000i
```

REAL $x[n]$ PROPERTY OF THE DFT

$$x[n] \text{ real} \quad \Longrightarrow \quad X[k] = X^*[\langle -k \rangle_N].$$

Equivalently,

$$x[n] \text{ real} \quad \Longrightarrow \quad X_r[k] = X_r[\langle -k \rangle_N] \quad \text{and} \quad X_i[k] = -X_i[\langle -k \rangle_N].$$

where $X_r[k]$ and $X_i[k]$ are the real and imaginary parts of $X[k]$.

Derivation:

$$\begin{aligned} x[n] \text{ real} \quad &\Longrightarrow \quad x[n] = x^*[n] \\ &\Longrightarrow \quad \text{DFT}\{x[n]\} = \text{DFT}\{x^*[n]\} \\ &\Longrightarrow \quad X[k] = X^*[\langle -k \rangle_N]. \end{aligned}$$

The following MATLAB code fragment illustrates the circular symmetries present in the DFT of a real signal. In this case, the signal is of odd length. Notice that $X_i(0)$ must be 0 because when k is 0, the relation $X[k] = X^*[\langle -k \rangle_N]$ gives $X_i[0] = -X_i[0]$.

```
>> x = [3 7 5 2 4 1 5]'

x =

     3
     7
     5
     2
     4
     1
     5
```

```
>> X = fft(x)

X =

  27.0000
   3.7409 - 4.5956i
  -1.3351 - 1.7780i
  -5.4058 + 4.2094i
  -5.4058 - 4.2094i
  -1.3351 + 1.7780i
   3.7409 + 4.5956i

>> [real(X) imag(X)]

ans =

   27.0000         0
    3.7409   -4.5956
   -1.3351   -1.7780
   -5.4058    4.2094
   -5.4058   -4.2094
   -1.3351    1.7780
    3.7409    4.5956
```

In the following example, the signal is of even length. Notice that $X_i(N/2) = 0$ in this case.

```
>> x = [3 7 5 2 4 1 3 2]'

x =

     3
     7
     5
```

```
     2
     4
     1
     3
     2

>> X = fft(x)

X =

  27.0000
   3.2426 - 6.2426i
  -1.0000 - 4.0000i
  -5.2426 - 2.2426i
   3.0000
  -5.2426 + 2.2426i
  -1.0000 + 4.0000i
   3.2426 + 6.2426i

>> [real(X) imag(X)]

ans =

   27.0000         0
    3.2426   -6.2426
   -1.0000   -4.0000
   -5.2426   -2.2426
    3.0000         0
   -5.2426    2.2426
   -1.0000    4.0000
    3.2426    6.2426
```

Real and Circularly Symmetric $x[n]$

When $x[n]$ is real and circularly symmetric, then so is the DFT.

$$\{x[n] \text{ real and } x[n] = x[\langle -n \rangle_N]\} \quad \Longrightarrow \quad \{X[k] \text{ real and } X[k] = X[\langle -k \rangle_N]\}$$

Derivation:

$$x[n] \text{ real} \quad \Longrightarrow \quad X[k] = X^*[\langle -k \rangle_N]$$

and

$$\begin{aligned} x[n] = x[\langle -n \rangle_N] \quad &\Longrightarrow \quad \text{DFT}\{x[n]\} = \text{DFT}\{x[\langle -n \rangle_N]\} \\ &\Longrightarrow \quad X[k] = X[\langle -k \rangle_N]. \end{aligned}$$

Therefore

$$\begin{aligned} \{x[n] \text{ real and } x[n] = x[\langle -n \rangle_N]\} \quad &\Longrightarrow \quad X[k] = X^*[\langle -k \rangle_N] = X[\langle -k \rangle_N] \\ &\Longrightarrow \quad X[k] \text{ real and } X[k] = X[\langle -k \rangle_N]. \end{aligned}$$

Real and Circularly Symmetric $x[n]$

The following example illustrates the DFT of a real circularly symmetric signal.

```
>> x = [3 1 5 4 6 4 5 1]'

x =

     3
     1
     5
     4
     6
     4
     5
     1

>> X = fft(x)

X =

  29.0000
  -7.2426 - 0.0000i
  -1.0000
   1.2426 - 0.0000i
   9.0000
   1.2426 + 0.0000i
  -1.0000
  -7.2426 + 0.0000i
```

Note that both $x[n]$ and $X[k]$ are real and circularly symmetric.

SUMMARY OF DFT PROPERTIES

Periodicity	$X[k] = X[\langle k \rangle_N]$	$x[n] = x[\langle n \rangle_N]$
Circular Shift	$x[\langle n - m \rangle_N]$	$W_N^{-mk} \cdot X[k]$
Modulation	$W_N^{mn} \cdot x[n]$	$X[\langle k - m \rangle_N]$
Circular Convolution	$x[n] \circledast g[n]$	$X[k] \cdot G[k]$
Time-Reversal	$x[\langle -n \rangle_N]$	$X[\langle -k \rangle_N]$
Complex Conjugation	$x^*[n]$	$X^*[\langle -k \rangle_N]$
Parseval's Theorem	$\sum_{n=0}^{N-1} x[n] \cdot g^*[n] = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X[k] \cdot G^*[k]$	