MINIMUM-PHASE FIR FILTER DESIGN
The Lifting Procedure

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INTRODUCTION

FIR filters that have symmetric or antisymmetric impulse responses have no phase distortion (their phase is linear). However, the problem with linear-phase filters is that the delay can be too large. The delay of a linear-phase filter is equal to $(N-1)/2$, where N is the length of the filter. To obtain a linear-phase FIR filter with a narrow transition-band and high stop-band attenuation requires making the filter long. Therefore, linear-phase FIR filters satisfying demanding specifications will have a large delay. This large delay could be a major drawback. (1) If the filter is used inside a feedback loop in a control system it could cause instability. (2) If the filter is used in a communication system it could cause delays that are longer than is acceptable.

For applications where it is important to minimize the delay caused by a filter, a minimum-phase filter can be a good choice. Minimum-phase filters have all their zeros inside or on the unit circle. A minimum-phase filter can be obtained from a linear-phase filter by reflecting all of the zeros that are outside the unit circle to inside the unit circle. In other words, those zeros located at $z = r\, e^{j\theta}$ are moved to $z = \frac{1}{r}\, e^{j\theta}$. This modification results in a minimum-phase filter that has the same frequency response magnitude $|H(e^{j\omega})|$ as the linear-phase filter. However, the minimum-phase filter obtained in this way will not be optimal in general. There will generally be a minimum-phase filter that is superior than the one obtained by simply reflecting the zeros in this way.

PROBLEM FORMULATION

In this section we consider the design of low-pass filters. The design problem can be stated as follows.

Problem 1: Find the FIR filter $h(n)$ of minimal length (not necessarily with linear-phase) such that

$$||H(e^{j\omega})| - 1| \le \Delta_p \qquad \text{for} \quad 0 \le \omega \le \omega_p \tag{1}$$

$$|H(e^{j\omega})| \le \Delta_s \qquad \text{for} \quad \omega_s \le \omega \le \pi. \tag{2}$$

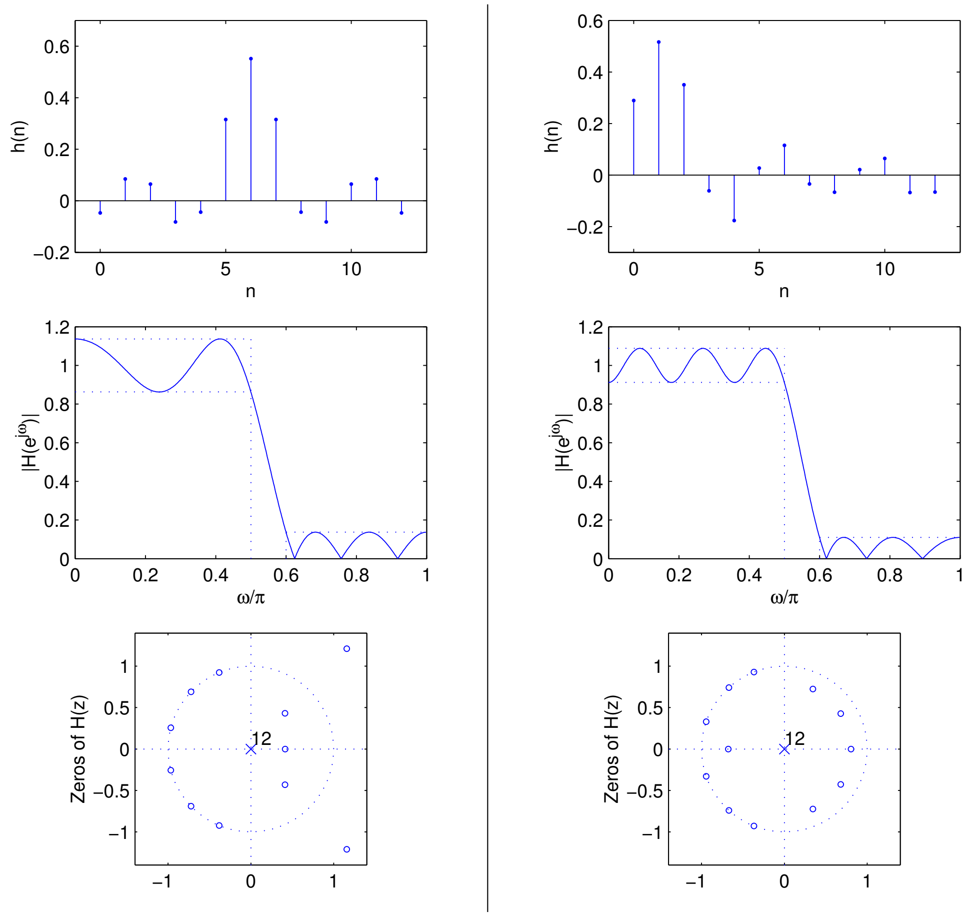
The given design parameters are the band-edges ω_p, ω_s and the maximum error in the pass-band and stop-band Δ_p, Δ_s.

This design problem is more difficult than the design of a linear-phase FIR filter, because the frequency response magnitude $|H(e^{j\omega})|$ is a nonlinear function of the filter coefficients $h(n)$.

$$|H(e^{j\omega})| = \left| \sum_{n=0}^{N-1} h(n)\, e^{j\omega n} \right|. \tag{3}$$

This section describes how to design minimum-phase FIR filters so that the error between $|H(e^{j\omega})|$ and a desired $D(\omega)$ is minimized.

EXAMPLE



The filter on the left is a linear-phase FIR filter of length 13 designed so that the Chebyshev error is minimized. The maximum value of the error in the pass-band and stop-band is 0.137. The filter on the right is a minimum-phase FIR filter of the same length, however, the maximum value of the error in the pass-band is 0.0882, and in the stop-band it is 0.1099. By not restricting the impulse response to be symmetric, the frequency response magnitude $|H(e^{j\omega})|$ can be improved. The remainder of this section describes how to design a non-symmetric low-pass filter $h(n)$ so that $|H(e^{j\omega})|$ has the smallest error.

THE SQUARE MAGNITUDE

Because $|H(e^{j\omega})|$ is not a linear function of the coefficients $h(n)$, it turns out to be more convenient to work with the square magnitude $|H(e^{j\omega})|^2$. Assuming the filter coefficients $h(n)$ are real, we can write:

$$|H(e^{j\omega})|^2 = H(e^{j\omega}) \cdot \overline{H(e^{j\omega})} \tag{4}$$

$$= H(e^{j\omega}) \cdot H(e^{-j\omega}) \tag{5}$$

$$= H(z) \cdot H(1/z)\big|_{z=e^{j\omega}} . \tag{6}$$

Suppose $h(n)$ is a length N FIR impulse response. If we define

$$R(z) := H(z) \cdot H(1/z) \tag{7}$$

then

$$r(n) = h(n) * h(-n) \tag{8}$$

and

$$R(e^{j\omega}) = |H(e^{j\omega})|^2. \tag{9}$$

Note that $r(n)$ will be symmetric with $r(n) = r(-n)$ and the length of $r(n)$ will be $2N - 1$. It turns out to be more straight-forward to design the filter $R(z)$, and then to obtain $H(z)$ from $R(z)$.

Herrmann and Schüssler proposed a method for designing minimum-phase filters for which the Chebyshev error is minimized [1]. Their method is based on the transformation of a linear-phase FIR Chebyshev filter in to a minimum-phase FIR filter.

Transformation of Type I FIR equi-ripple linear-phase filter into an equi-ripple minimum-phase filter

The method consists of 4 steps.

1. Design an equi-ripple Type I linear-phase filter $h_1(n)$ of length $2N-1$.

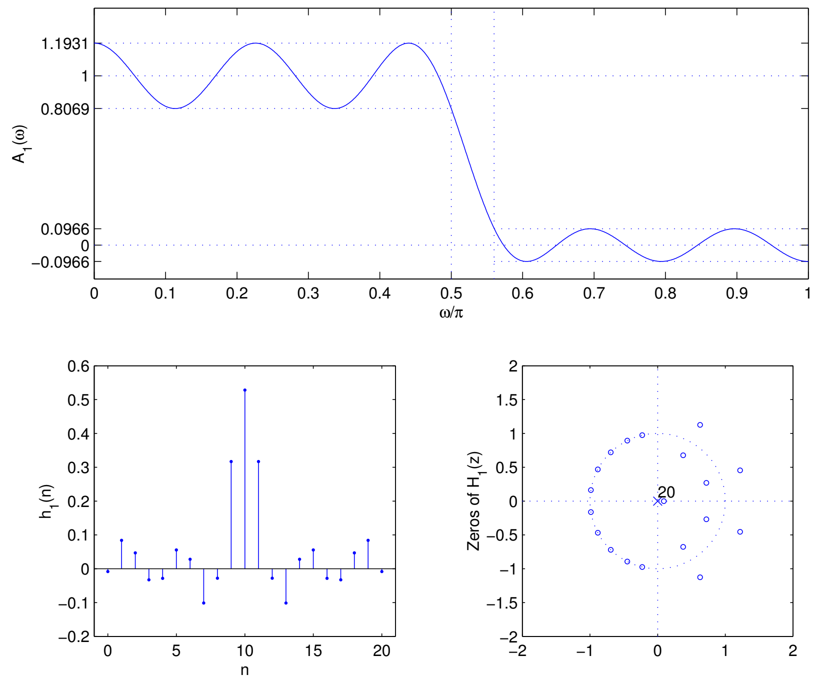
2. Obtain a second Type I linear-phase filter $h_2(n)$ by adjusting $h_1(n)$ so that $A_2(\omega) \geq 0$.

3. Spectrally factor $h_2(n)$ to obtain a minimum-phase FIR filter $h_3(n)$ of length N.

4. Scale $h_3(n)$ by an appropriate constant to obtain a minimum-phase FIR filter $h(n)$ that is equi-ripple.

This procedure is best illustrated by an example.

Step One: Design an equi-ripple Type I FIR filter of length $2N-1$. For example, to design a length 11 minimum-phase FIR filter, we will first design a length 21 Type I FIR filter.



Define δ_1 and δ_2 to be the maximum value of the error in the pass-band and stop-band.

$$\delta_1 := \max |A_1(\omega) - 1| \qquad \text{for} \quad 0 \le \omega \le \omega_p \tag{10}$$

$$\delta_2 := \max |A_1(\omega)| \qquad \text{for} \quad \omega_s \le \omega \le \pi \tag{11}$$

We can then write:

$$1 - \delta_1 \le A_1(\omega) \le 1 + \delta_1 \qquad \text{for} \quad 0 \le \omega \le \omega_p \tag{12}$$

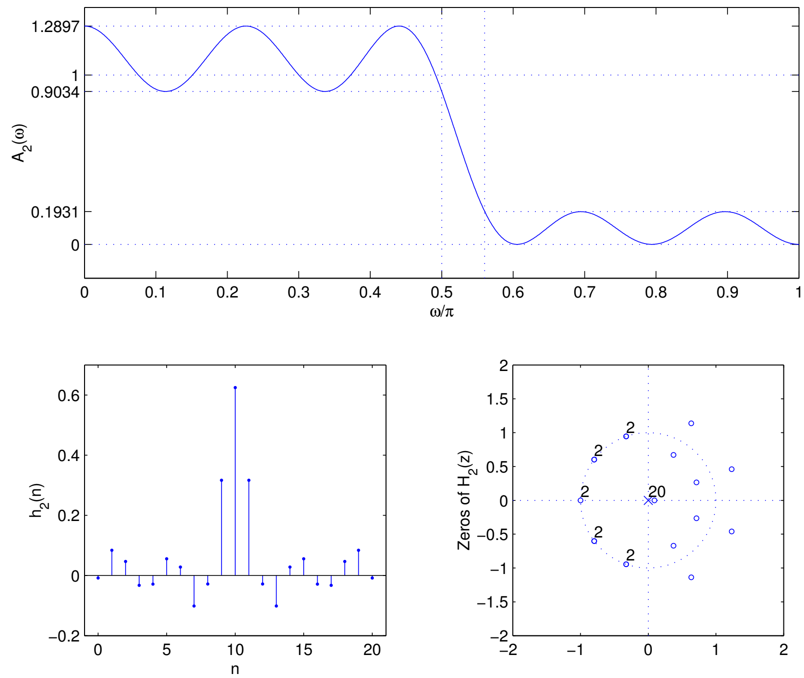
$$-\delta_2 \le A_1(\omega) \le \delta_2 \qquad \text{for} \quad \omega_s \le \omega \le \pi. \tag{13}$$

```
N  = 11;
Kp = 1;
Ks = 2;
wp = 0.50*pi;
ws = 0.56*pi;
wo = (wp+ws)/2;
L  = 1000;
w  = [0:L]*pi/L;
D  = (w<=wo);
W  = Kp*(w<=wp) + Ks*(w>=ws);
[h1,del] = fircheb(2*N-1,D,W);
d1 = del/Kp;
d2 = del/Ks;
```

In this example, $\delta_1 = 0.1931$, $\delta_2 = 0.0966$.

STEP TWO

Step Two: Step two is to adjust the filter $h_1(n)$ so that the filter can be spectrally factored. This can be done by adding δ_2 to the middle coefficient of $h_1(n)$.



$$\boxed{h_2(n) = h_1(n) + \delta_2 \cdot \delta(n - N + 1)} \tag{14}$$

In Matlab this can be done with the command

```
h2 = h1;
h2(N) = h2(N) + d2;
```

This has the effect of 'lifting' the amplitude response

$$\boxed{A_2(\omega) = A_1(\omega) + \delta_2.} \tag{15}$$

The frequency response amplitude of $h_2(n)$ then satisfies the following bounds,

$$1-\delta_1+\delta_2 \le A_2(\omega) \le 1+\delta_1+\delta_2 \qquad \text{for} \quad 0 \le \omega \le \omega_p \tag{16}$$

$$0 \le A_2(\omega) \le 2\,\delta_2 \qquad \text{for} \quad \omega_s \le \omega \le \pi. \tag{17}$$

Notice that the zeros of $H_2(z)$ on the unit circle are now of even multiplicity. Then the filter $h_2(n)$ can be spectrally factored because

$$\boxed{A_2(\omega) \ge 0} \tag{18}$$

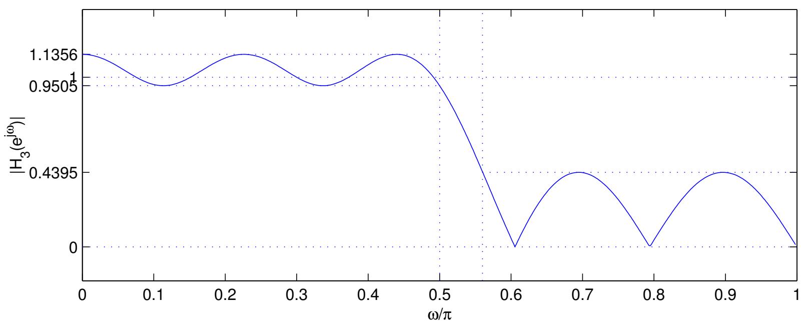
and because $h_2(n)$ is symmetric and of odd length.

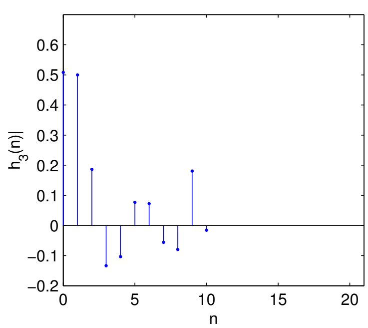
Step Three: We can obtain a minimum-phase filter by spectrally factoring the filter $h_2(n)$. If $h_3(n)$ is a spectral factor of $h_2(n)$, then we have

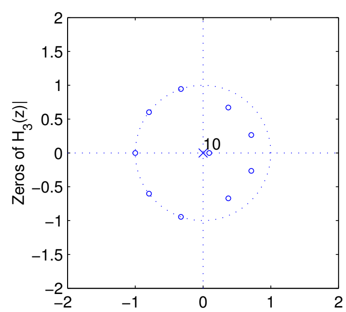
$$|H_3(e^{j\omega})|^2 = A_2(\omega). \tag{19}$$

or

$$\boxed{|H_3(e^{j\omega})| = \sqrt{A_2(\omega)}.} \tag{20}$$







This can be done with the Matlab command,

```
h3 = sfact(h2);
```

where the command `sfact` is described in the notes about spectral factorization. Then $|H_3(e^{j\omega})|$ satisfies the following bounds

$$\sqrt{1-\delta_1+\delta_2} \le |H_3(e^{j\omega})| \le \sqrt{1+\delta_1+\delta_2} \qquad \text{for} \quad 0 \le \omega \le \omega_p \tag{21}$$

$$|H_3(e^{j\omega})| \le \sqrt{2\,\delta_2} \qquad \text{for} \quad \omega_s \le \omega \le \pi. \tag{22}$$

Note that $|H_3(e^{j\omega})|$ does not have an equi-ripple behavior in the pass-band. That is because the lower bound $\sqrt{1-\delta_1+\delta_2}$ and the upper bound $\sqrt{1+\delta_1+\delta_2}$ in the pass-band are not of the form $1+\delta_p$ and $1-\delta_p$.

Step Four: Finally, we need only scale $h_3(n)$ so that it has an equi-ripple behavior in the pass-band. The scaling

$$h(n) = C \cdot h_3(n), \qquad |H(e^{j\omega})| = C \cdot |H_3(e^{j\omega})| \tag{23}$$

gives the bounds

$$C\,\sqrt{1 - \delta_1 + \delta_2} \le |H(e^{j\omega})| \le C\,\sqrt{1 + \delta_1 + \delta_2} \qquad \text{for} \quad 0 \le \omega \le \omega_p \tag{24}$$

$$|H(e^{j\omega})| \le C\,\sqrt{2\,\delta_2} \qquad \text{for} \quad \omega_s \le \omega \le \pi. \tag{25}$$

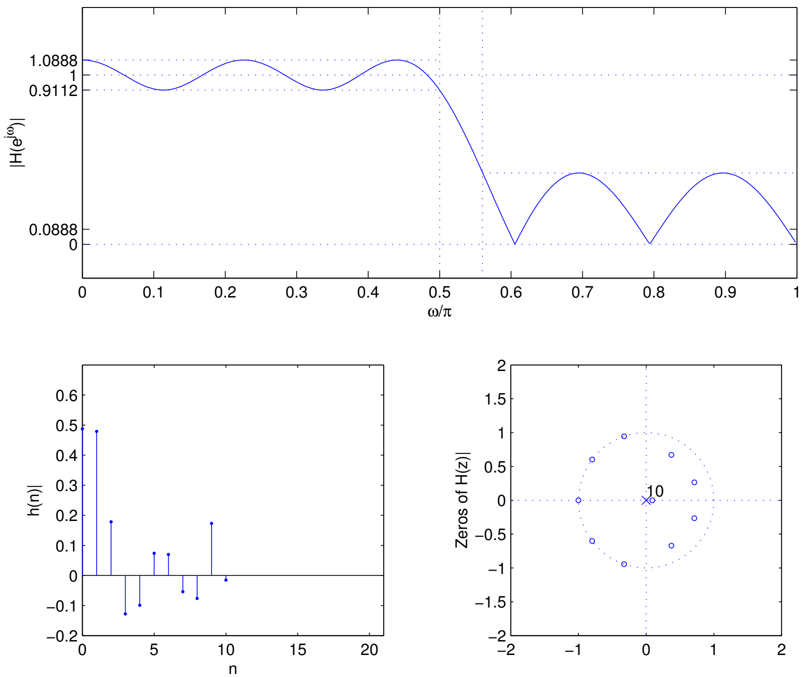
Setting the lower and upper bounds equal to $1 - \delta_p$ and $1 + \delta_p$ respectively, we get

$$1 - \delta_p = C\,\sqrt{1 - \delta_1 + \delta_2} \tag{26}$$

$$1 + \delta_p = C\,\sqrt{1 + \delta_1 + \delta_2}. \tag{27}$$

Solving for C, we obtain

$$\boxed{C = \frac{2}{\sqrt{1 + \delta_1 + \delta_2} + \sqrt{1 - \delta_1 + \delta_2}}.} \tag{28}$$



In Matlab we have

```
h  = h3*2/(sqrt(1+d1+d2)+sqrt(1-d1+d2));
```

With this value of C, we get the bounds

$$1 - \delta_p \le |H(e^{j\omega})| \le 1 + \delta_p \qquad \text{for} \quad 0 \le \omega \le \omega_p \tag{29}$$

$$|H(e^{j\omega})| \le \delta_s \qquad \text{for} \quad \omega_s \le \omega \le \pi. \tag{30}$$

where

$$\delta_p = \frac{\sqrt{1+\delta_1+\delta_2} - \sqrt{1-\delta_1+\delta_2}}{\sqrt{1+\delta_1+\delta_2} + \sqrt{1-\delta_1+\delta_2}} \tag{31}$$

$$\delta_s = \frac{2\sqrt{2\,\delta_2}}{\sqrt{1+\delta_1+\delta_2} + \sqrt{1-\delta_1+\delta_2}}. \tag{32}$$

To ensure that $H(e^{j\omega})$ satisfies the given error bound we can invert the expressions for δ_p and δ_s to determine the corresponding target values δ_1, δ_2. Solving for δ_1 and δ_2 gives

$$\delta_1 = \frac{4\,\delta_p}{2 - \delta_s^2 + 2\,\delta_p^2} \tag{33}$$

$$\delta_2 = \frac{\delta_s^2}{2 - \delta_s^2 + 2\,\delta_p^2}. \tag{34}$$

Then, provided the linear-phase Type I filter $h_1(n)$ is designed according to the tolerances δ_1 and δ_2, the minimum-phase filter $h(n)$ will satisfy the constraints

$$||H(e^{j\omega})| - 1| \le \delta_p \qquad \text{for} \quad 0 \le \omega \le \omega_p \tag{35}$$

$$|H(e^{j\omega})| \le \delta_s \qquad \text{for} \quad \omega_s \le \omega \le \pi. \tag{36}$$

Note that a Type I FIR filter specified to have a maximum error of Δ_1 and Δ_2 in the pass-band and stop-band respectively, can be designed using the Parks-McClellan (PM) algorithm with weights $K_p = 1/\Delta_1$, $K_s = 1/\Delta_2$. However, the resulting Type I FIR produced by the PM algorithm may have errors that are smaller than the specified tolerances. Therefore, the formulas (33,34) are to be used only to select the weighting parameters K_p and K_s.
The Matlab code for this example is combined into the following program.

```
N  = 11;
Kp = 1;
Ks = 2;
wp = 0.50*pi;
ws = 0.56*pi;
wo = (wp+ws)/2;
L  = 3000;
```

```
w  = [0:L]*pi/L;
D  = (w<=wo);
W  = Kp*(w<=wp) + Ks*(w>=ws);
[h1,del] = fircheb(2*N-1,D,W);
d1 = del/Kp;
d2 = del/Ks;

h2 = h1;
h2(N) = h2(N) + d2;

h3 = sfact(h2);

A  = sqrt(1+d1+d2);
B  = sqrt(1-d1+d2);
h  = h3*2/(A+B);

dp = (A-B)/(A+B);
ds = 2*sqrt(2*d2)/(A+B);
```

SUMMARY OF THE LIFTING PROCEDURE

To design a minimum-phase filter satisfying

$$||H(e^{j\omega})| - 1| \le \Delta_p \qquad \text{for} \quad 0 \le \omega \le \omega_p \tag{37}$$

$$|H(e^{j\omega})| \le \Delta_s \qquad \text{for} \quad \omega_s \le \omega \le \pi \tag{38}$$

the following algorithm can be used. This algorithm is sometimes referred to as the *lifting* procedure as it involves lifting the amplitude response $A_1(\omega)$ so that it can be spectrally factored.

1. Design a minimal-length Type I FIR filter $h_1(n)$ satisfying the constraints

   $$|A_1(\omega) - 1| \le \Delta_1 \qquad \text{for} \quad 0 \le \omega \le \omega_p \tag{39}$$

   $$|A_1(\omega)| \le \Delta_2 \qquad \text{for} \quad \omega_s \le \omega \le \pi. \tag{40}$$

   where

   $$\Delta_1 = \frac{4\,\Delta_p}{2 - \Delta_s^2 + 2\,\Delta_p^2} \tag{41}$$

   $$\Delta_2 = \frac{\Delta_s^2}{2 - \Delta_s^2 + 2\,\Delta_p^2}. \tag{42}$$

   This filter can be obtained with the Parks-McClellan algorithm using the pass-band and stop-band weights

   $$K_p = 1/\Delta_1, \qquad K_s = 1/\Delta_2. \tag{43}$$

2. Define

   $$\delta_1 := \max |A_1(\omega) - 1| \qquad \text{for} \quad 0 \le \omega \le \omega_p \tag{44}$$

   $$\delta_2 := \max |A_1(\omega)| \qquad \text{for} \quad \omega_s \le \omega \le \pi \tag{45}$$

and set

$$h_2(n) = h_1(n) + \delta_2 \cdot \delta(n - N + 1) \tag{46}$$

3. Compute a minimum-phase spectral factor $h_3(n)$ of $h_2(n)$ such that

$$|H_3(e^{j\omega})| = \sqrt{|H_2(e^{j\omega})|}. \tag{47}$$

4. Scale $h_3(n)$ to obtain the equi-ripple filter $h(n)$.

$$h(n) = C \cdot h_3(n) \tag{48}$$

where

$$C = \frac{2}{\sqrt{1 + \delta_1 + \delta_2} + \sqrt{1 - \delta_1 + \delta_2}}. \tag{49}$$

Note that when the spectral factorization is performed,

$$|H_3(e^{j\omega})| = \sqrt{|H_2(e^{j\omega})|} \tag{50}$$

the stop-band ripples become larger. If the spectral factor $h_3(n)$ is to have small ripples in the stop-band, the filter $h_2(n)$ must have ripples in the stop-band that are especially small.

In this example, we use the lifting procedure to design a minimum-phase FIR filter satisfying the specifications

$$||H(e^{j\omega})| - 1| \le \Delta_p \qquad \text{for} \quad 0 \le \omega \le \omega_p \tag{51}$$

$$|H(e^{j\omega})| \le \Delta_s \qquad \text{for} \quad \omega_s \le \omega \le \pi \tag{52}$$

where

$$\Delta_p = 0.03, \quad \Delta_s = 0.02, \quad \text{and} \quad \omega_p = 0.50\pi, \quad \omega_s = 0.56\pi. \tag{53}$$

Step one. Step one of the design processes calls for the design of a Type I linear-phase FIR filter meeting the transformed specifications. The transformed specifications for this example are

$$\Delta_1 = 0.0600, \quad \Delta_2 = 1.9986 \cdot 10^{-4}. \tag{54}$$

Therefore the weighting parameters K_p and K_s can be chosen to be

$$K_p = 16.6783, \qquad K_s = 5.0035 \cdot 10^3. \tag{55}$$

Notice that the stop-band is weighted much more than the pass-band. If we design a length 79 Type I FIR filter with these weighting parameters and the specified pass-band and stop-band edges, we find that the resulting pass-band and stop-band errors are

$$\delta_1 = 0.0605, \qquad \delta_2 = 2.0177 \cdot 10^{-4} \tag{56}$$

which violated the specified tolerance margins Δ_1 and Δ_2. But if we design a length 81 Type I FIR filter with the same weighting parameters, then we find that

$$\delta_1 = 0.0571, \qquad \delta_2 = 1.9032 \cdot 10^{-4} \tag{57}$$

which satisfy the transformed specifications. We call this filter $h_1(n)$.

Step two. In step two, we just need to add δ_2 to the center term of $h_1(n)$ and call the result $h_2(n)$. Then the zeros of $H_2(z)$ that lie on the unit circle will be of even multiplicity.

Step three. In step three we find the minimum-phase spectral factor of $h_2(n)$ and call the result $h_3(n)$. The length of the $h_3(n)$ will be 41.

Step four. In step three we scale the filter $h_3(n)$ by C, which for this example is $C = 1.0003127316$.

This solution found using the lifting procedure yields a minimum-phase FIR filter of length 41 with

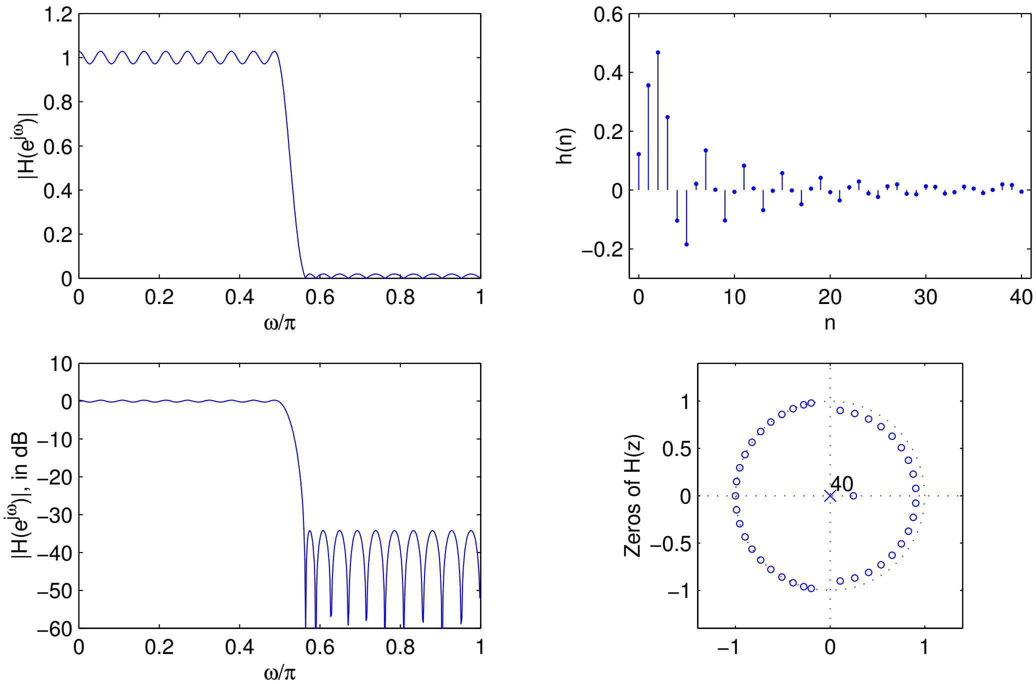
$$||H(e^{j\omega})| - 1| \leq \delta_p \qquad \text{for} \quad 0 \leq \omega \leq \omega_p \tag{58}$$

$$|H(e^{j\omega})| \leq \delta_s \qquad \text{for} \quad \omega_s \leq \omega \leq \pi \tag{59}$$

where

$$\delta_p = 0.0286, \qquad \delta_s = 0.0195. \tag{60}$$

The minimum-phase filter is illustrated in the figure.

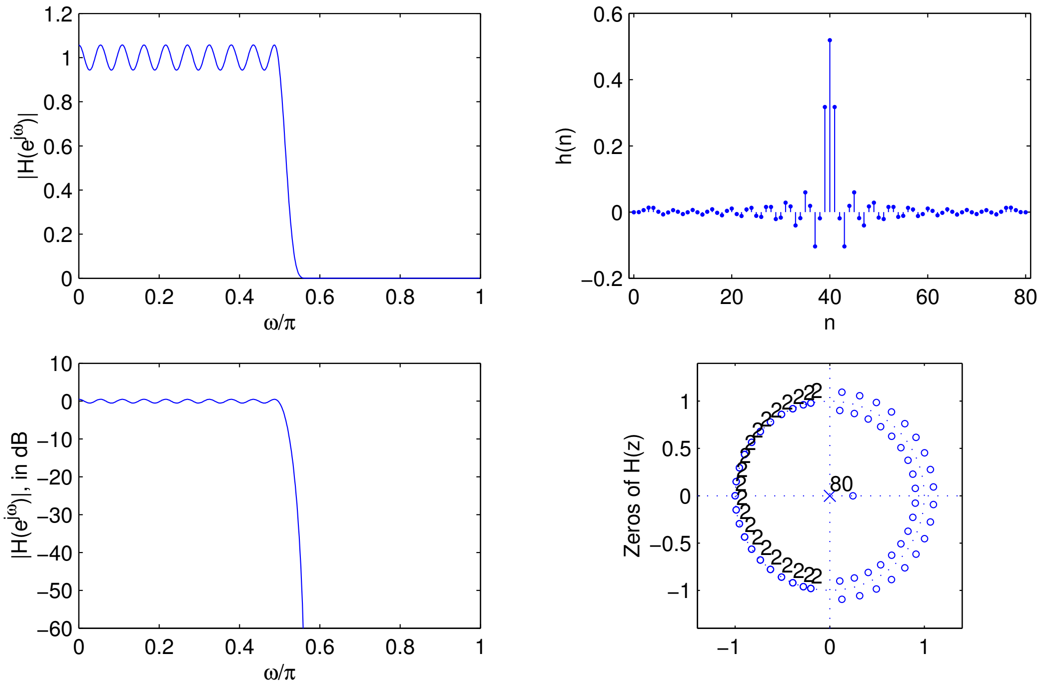


The filter was obtained with the following Matlab code.

```
% Set up specifications

Dp = 0.03;                            % Dp: pass-band spec for h
Ds = 0.02;                            % Ds: stop-band spec for h
D1 = (4*Dp)/(2-Ds^2 + 2*Dp^2);        % D1: pass-band spec for h1
D2 = (Ds^2)/(2-Ds^2 + 2*Dp^2);        % D2: stop-band spec for h1
Kp = 1/D1;                            % Kp: pass-band weight for h1
Ks = 1/D2;                            % Ks: stop-band weight for h1
wp = 0.50*pi;
ws = 0.56*pi;
wo = (wp+ws)/2;
L  = 3000;
w  = [0:L]*pi/L;
D  = (w<=wo);
W  = Kp*(w<=wp) + Ks*(w>=ws);

% -----------------------------------------------------------

% STEP 1 - Design a Type I FIR filter of minimal length that
%          meets the transformed specifications.

N  = 40;
[h1,del] = fircheb(2*N-1,D,W);
d1 = del/Kp;                         % d1: actual pass-band error for h1
d2 = del/Ks;                         % d2: actual stop-band error for h1

[d1 d2] % ---> A length 79 filter can not satisfy the specifications.
```

```
% Try a filter of length 81

N  = 41;
[h1,del] = fircheb(2*N-1,D,W);
d1 = del/Kp;                    % d1: actual pass-band error for h1
d2 = del/Ks;                    % d2: actual stop-band error for h1

[d1 d2] % ---> This length 81 filter does satisfy the specifications.

figure(1), mfigNS(h1), orient tall, print -depsc lift2A

% -----------------------------------------------------------

% STEP 2 - "lift" the frequency response

h2 = h1;
h2(N) = h2(N) + d2;             % "lift" h1

figure(2), mfigNS(h2), orient tall, print -depsc lift2B

% -----------------------------------------------------------

% STEP 3 - Compute a minimum-phase spectral factor

h3 = sfact(h2);                 % spectral factorization of h2

figure(3), mfigNS(h3), orient tall, print -depsc lift2C

% -----------------------------------------------------------

% STEP 4 - Scale h(n)

A  = sqrt(1+d1+d2);
B  = sqrt(1-d1+d2);
h  = h3*2/(A+B);                % scaling of h3

figure(4), mfigNS(h), orient tall, print -depsc lift2D

% -----------------------------------------------------------

dp = (A-B)/(A+B);               % actual pass-band error for h
ds = 2*sqrt(2*d2)/(A+B);        % actual stop-band error for h
```

To compare the minimum-phase to a linear-phase filter with the same length, we obtain the linear-phase filter shown in the figure.



The pass-band and stop-band errors for the linear-phase filter are:

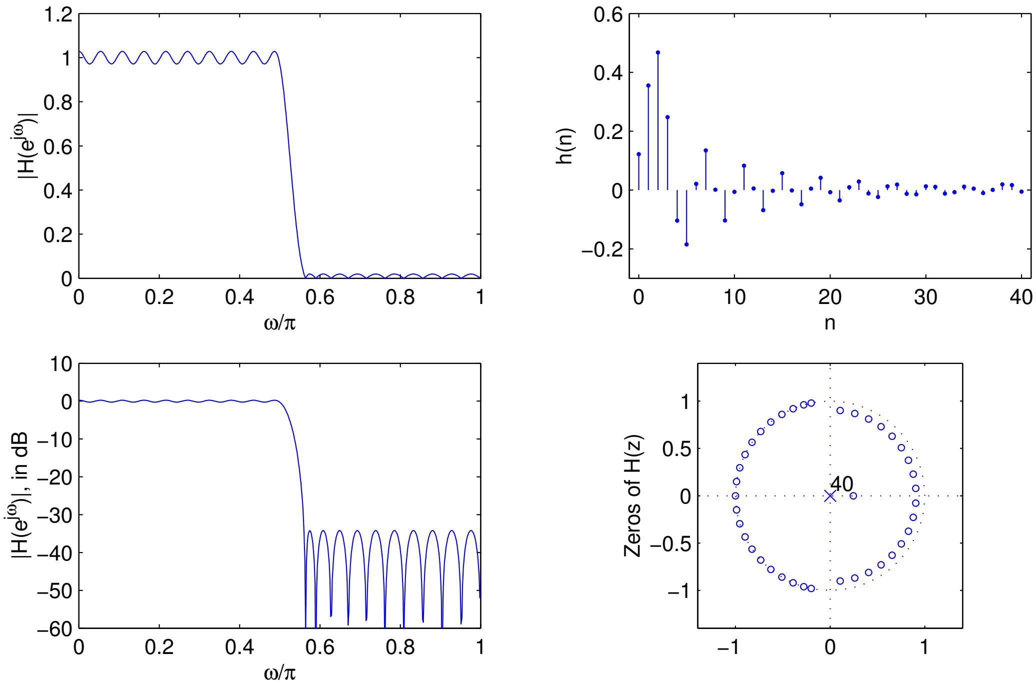
$$\delta_p = 0.0461, \qquad \delta_s = 0.0307. \tag{61}$$

which are greater than the those of the minimum-phase solution. This linear-phase filter was obtained with the following Matlab commands.

```
N  = 41;
Dp = 0.03;
Ds = 0.02;
Kp = 1/Dp;
Ks = 1/Ds;
wp = 0.50*pi;
ws = 0.56*pi;
wo = (wp+ws)/2;
L  = 3000;
w  = [0:L]*pi/L;
D  = (w<=wo);
W  = Kp*(w<=wp) + Ks*(w>=ws);
[h,del] = fircheb(N,D,W);
dp = del/Kp;
ds = del/Ks;
```

Therefore, we observe that by using a minimum-phase filter obtained using the lifting procedure, we get a filter that has (1) a better frequency response magnitude and (2) a shorter delay. Notice that the impulse response of the minimum-phase filter has most of its energy in the first few samples. The trade-off is that the minimum-phase filter introduces phase-distortion, while the linear-phase filter does not. The relative importance of delay versus phase-distortion depends on the application. In image processing for example, it is usually important to use filters that have symmetric impulse responses. When a filter with a non-symmetric impulse response is used in image processing it can causes distortion that is visible (around edges in particular). However, in other applications, the phase-distortion is less important. For example, in audio processing and compression, minimum-phase filters are often used because the ear is less sensitive to phase distortion.

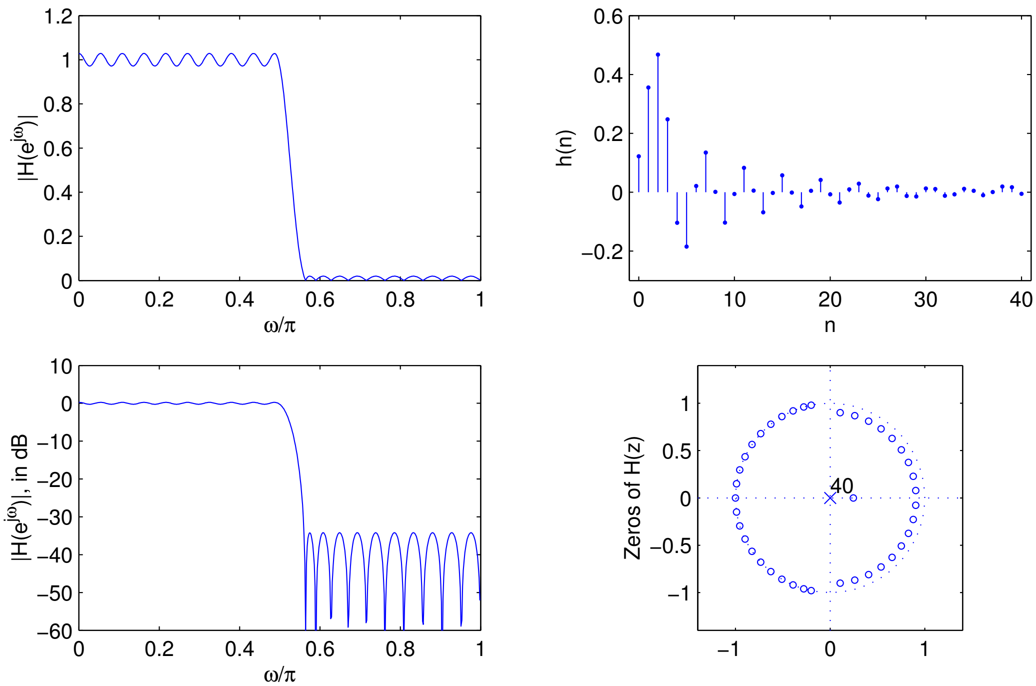
It was mentioned in the introduction that a minimum-phase filter can always be obtained by reflecting zeros z_k that lie outside the unit circle to $1/z_k$. If this is done with the length 41 linear-phase in the previous figure, one obtains the following filter.



The delay is reduced, but the frequency response magnitude $|H_1(e^{j\omega})|$ is not changed, and is therefore not optimal. This minimum-phase filter was obtained with the following Matlab commands.

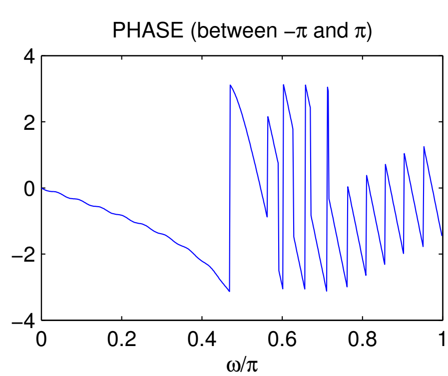
```
r = roots(h);
k = abs(r)>1.01;
r1 = r(k);          % roots outside |z|=1
r2 = r(~k);         % roots on or inside |z|=1
s = [1./r1; r2];
h2 = poly(leja(s));
h2 = h2*norm(h)/norm(h2);
```

Let us examine the phase of the minimum-phase filter we obtained earlier (illustrated here again for convenience).



We can compute the complex-valued frequency response with the the `freqz` command. Then the phase can be obtained with the `angle` command.

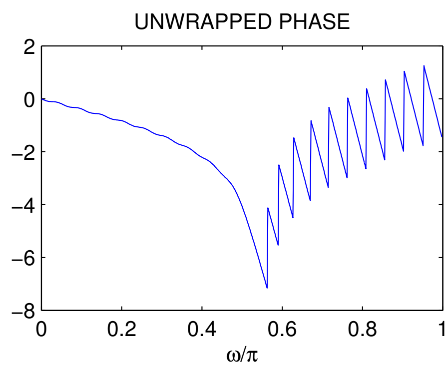
```
[H,w] = freqz(h);
plot(w/pi,angle(H))
```



PHASE UNWRAPPING

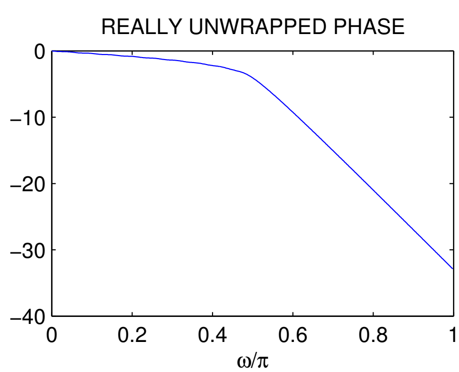
The plot of the phase has many discontinuities because the angle of a complex number provided by the `angle` command is between $-\pi$ and π. That causes jumps of size 2π. The processes of removing those jumps is called *phase unwrapping* and is implemented with the command `unwrap`.

```
plot(w/pi,unwrap(angle(H)))
```



There are still some jumps of size π corresponding to zeros on the unit circle. These can be removed by multiplying the phase by 2, using the `unwrap` command, and then dividing the result by 2.

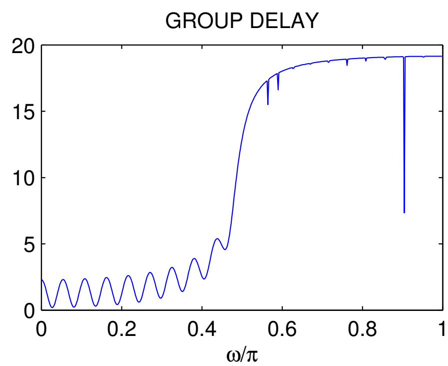
```
plot(w/pi,unwrap(angle(H)*2)/2)
```



GROUP DELAY

The *group delay* of a filter is the negative derivative of the phase. It indicates the delay caused by the filter measured in samples. It can be computed with the Matlab command `grpdelay`.

```
[G,w] = grpdelay(h);
```



Because the value of the magnitude is very small in the stop-band (ideally, it is zero), the phase and group delay in the stop-band is actually not so important.

References

[1] O. Herrmann and H. W. Schuessler. Design of nonrecursive filters with minimum phase. *Electron. Lett.*, 6(11):329–330, 28th May 1970. Also in [2].

[2] L. R. Rabiner and C. M. Rader, editors. *Digital Signal Processing*. IEEE Press, 1972.